

Braids, conformal module and entropy

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Abstract

In the paper we discuss two invariants of conjugacy classes of braids. The first invariant is the conformal module which occurred in connection with the interest in the 13th Hilbert problem. The second is a popular dynamical invariant, the entropy. It occurred in connection with Thurston's theory of surface homeomorphisms.

We prove that these invariants are related: They are inverse proportional. This allows to use known results on entropy for applications to the concept of conformal module. The conformal module provides an obstruction for isotopy of continuous objects involving braids to the respective holomorphic objects. We give applications for the case of quasipolynomials of degree three as well as for elliptic fiber bundles. Further, we give a short conceptual proof of a theorem which appeared in connection with research on the Thirteen's Hilbert Problem.

A byproduct of the proof is a systematic treatment of reducible braids and of the entropy of mapping classes on Riemann surfaces of second kind, as well as expressions of entropy and conformal module of conjugacy classes of reducible braids in terms of the respective invariants of the irreducible components.

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CHAPTER 1

Introduction

Braids play a role in several mathematical fields. It is therefore not surprising that braid invariants occurred in different areas of mathematics. We consider two invariants of conjugacy classes of braids, an invariant related to algebraic geometry, the conformal module, and an invariant related to topological dynamics, the entropy.

We will discuss the role of these invariants and give new applications of the concept of the conformal module. This invariant has been invented and used (without name) in connection with research initiated by the 13th Hilbert problem. Our main result is Theorem 1 which relates the two invariants. This allows to use results obtained for the nowadays popular entropy to get information about the conformal module. Vice versa, in a forthcoming paper we will show that methods of quasi-conformal mappings will give information about the conformal module which has new corollaries for the entropy.

In the introduction we give only an outline of the main purpose of the paper. For further results and details we refer to chapters 3 to 9. We added a chapter which contains recollections of notions and results in different topics (see chapter 2). In particular, some notions used in the introduction will be explained there with some details. The results collected in chapter 2 are used to prove the theorems of the present paper. Chapter 2 is added for convenience of the reader who is not familiar with some of the collected facts and notions, but mainly, to make the relation between known results in different topics transparent and to keep the notation consistent. Chapter 2 also contains some facts which are less commonly known and prepare the proof of the theorems of the paper.

We give now a brief description of the invariants and the related concepts.

Taking the point of view of **algebraic geometry** we consider braids on n strands as isotopy classes of geometric braids or, in slightly different words, as isotopy classes of continuous mappings γ of the interval $[0, 1]$ to the space of unordered n -tuples of points in the complex plane \mathbb{C} , with fixed base point $\gamma(0) = \gamma(1)$. Alternatively, braids on n strands are given by words in the Artin group \mathcal{B}_n .

Denote by \mathfrak{P}_n the space of monic polynomials of degree n without multiple zeros. This space can be parametrized either by the coefficients or by the unordered tuple of zeros of polynomials. This makes \mathfrak{P}_n a complex manifold, in fact, the complement of the algebraic hypersurface $\{D_n = 0\}$ in complex Euclidean space \mathbb{C}^n . Here D_n denotes the discriminant of polynomials in \mathfrak{P}_n . The function D_n is a polynomial in the coefficients of elements of \mathfrak{P}_n . Arnol'd studied the topological invariants of the space \mathfrak{P}_n ([4]). Choose a base point $E_n \in \mathfrak{P}_n$. Using the second parametrization Arnol'd interpreted the group \mathcal{B}_n of n -braids as elements of the fundamental group $\pi_1(\mathfrak{P}_n, E_n)$ with base point E_n .

The conjugacy classes $\hat{\mathcal{B}}_n$ of the braid group, equivalently, of the fundamental group $\pi_1(\mathfrak{P}_n, E_n)$, can be interpreted as free homotopy classes of loops in \mathfrak{P}_n . We define the following collection of conformal invariants of the complex manifold \mathfrak{P}_n . Consider an element \hat{b} of $\hat{\mathcal{B}}_n$. We say that a continuous mapping f of an annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$, $0 \leq r \leq R$, into \mathfrak{P}_n represents \hat{b} if for some (and hence for any) circle $\{|z| = \rho\} \subset A$ the loop $f : \{|z| = \rho\} \rightarrow \mathfrak{P}_n$ represents \hat{b} .

Ahlfors defined the conformal module of an annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$ as $m(A) = \frac{1}{2\pi} \log(\frac{R}{r})$. Two annuli of finite conformal module are conformally equivalent iff they have equal conformal module. If a manifold Ω is conformally equivalent to an annulus A , its conformal module is defined to be $m(A)$. Recall that any domain in the complex plane with fundamental group isomorphic to the group of integer numbers \mathbb{Z} is conformally equivalent to an annulus.

Associate with each conjugacy class of the fundamental group of \mathfrak{P}_n , or, equivalently, with each conjugacy class of n -braids, its conformal module defined as follows.

Definition 1. *Let \hat{b} be a conjugacy class of n -braids, $n \geq 2$. The conformal module $M(\hat{b})$ of \hat{b} is defined as $M(\hat{b}) = \sup_{A(\hat{b})} m(A)$, where $A(\hat{b})$ denotes the set of all annuli which admit a holomorphic mapping into \mathfrak{P}_n which represents \hat{b} .*

The conformal module serves as obstruction for continuous or smooth objects involving braids to the respective holomorphic objects.

Runge's approximation theorem shows that the conformal module is positive for any conjugacy class of braids.

For any complex manifold the conformal module of conjugacy classes of its fundamental group can be defined. The collection of conformal modules of all conjugacy classes is a biholomorphic invariant of the manifold. This concept seems to be especially useful for locally symmetric spaces, for instance, for the quotient of the n -dimensional round complex ball by a subgroup of its automorphism group which acts freely and properly discontinuously. In this case the universal covering is the ball, and the fundamental group of the quotient manifold can be identified with the group of covering translations. For each covering translation the problem is to consider the quotient of the ball by the action of the group generated by this single covering translation and to maximize the conformal module of annuli which admit holomorphic mappings into this quotient. The latter concept can be generalized to general mapping class groups. The generalization has relations to symplectic fiber bundles.

It will also be interesting to consider the conformal module of conjugacy classes of the fundamental group of complements of complex hypersurfaces in \mathbb{P}^n . Note that Zariski initiated the study of fundamental groups of such complements, giving special attention to the case $n = 2$ (see [39], [40]).

The concept of the conformal module of conjugacy classes of braids appeared (without name) in the paper [15] which was motivated by the interest of the authors in Hilbert's Thirteen's Problem for algebraic functions. In chapter 8 we will give a short comment on the role of the Thirteen's Hilbert Problem in initiating the research of several mathematicians.

The following objects related to \mathfrak{P}_n have been considered in this connection. A continuous mapping from a topological space X into the set of monic

polynomials of fixed degree (maybe, with multiple zeros) is a quasipolynomial. It can be written as function in two variables $x \in X$, $\zeta \in \mathbb{C}$: $f(x, \zeta) = a_0(x) + a_1(x)\zeta + \dots + a_{n-1}(x)\zeta^{n-1} + \zeta^n$, for continuous functions a_j , $j = 1, \dots, n$, on X . If X is a complex manifold and the mapping is holomorphic it is called an algebroid function. If the image of the map is contained in \mathfrak{P}_n it is called separable. A separable quasipolynomial is called solvable if it can be globally written as a product of quasipolynomials of degree 1, and is called irreducible if it can not be written as product of two quasipolynomials of positive degree. Two separable quasipolynomials are isotopic if there is a continuous family of separable quasipolynomials joining them. An algebroid function on \mathbb{C}^n whose coefficients are polynomials is called an algebraic function. Let X be a complex manifold. For a separable quasipolynomial, considered as a function f on $X \times \mathbb{C}$, its zero set $\mathfrak{S}_f = \{(x, \zeta) \in X \times \mathbb{C}, f(x, \zeta) = 0\}$ is a symplectic surface, called braided surface due to its relation to braids.

It will be useful to have the following result and terminology in mind. Let X be a connected open Riemann surface of finite genus with at most countably many ends. By [17] X is conformally equivalent to a domain Ω on a closed Riemann surface X^c such that the connected components of $X^c \setminus \Omega$ are all points or closed geometric discs. A geometric disc is a topological disc whose lift to the universal covering is a round disc (in the standard metric of the covering). If all connected components of $X^c \setminus \Omega$ are points then X is called of first kind, otherwise it is called of second kind. A Riemann surface is called finite if its fundamental group is finitely generated.

The conformal module of conjugacy classes of braids serves as obstruction for the existence of isotopies of quasipolynomials (respectively, of braided surfaces) to algebroid functions (respectively, to complex curves). Indeed, let X be an open Riemann surface. Suppose f is a separable quasipolynomial of degree n on X . Consider any domain $A \subset X$ which is conformally equivalent to an annulus. The restriction of f to A defines a mapping of the domain A into the space of polynomials \mathfrak{P}_n , hence it defines a conjugacy class of n -braids $\hat{b}_{f,A}$. The following lemma is obvious.

Lemma 1. *If f is algebroid then $m(A) \leq M(\hat{b}_{f,A})$.*

For the **dynamical concept** braids on n strands are considered as elements of the mapping class group of the n -punctured disc. Let \mathbb{D} be the unit disc in the complex plane. Denote by E_n^0 the set consisting of the n points $0, \frac{1}{n}, \dots, \frac{n-1}{n}$. Consider homeomorphisms of the n -punctured disc $\mathbb{D} \setminus E_n^0$, which fix the boundary $\partial\mathbb{D}$ pointwise. Equivalently, these are the mappings which extend to homeomorphisms of the closed disc \mathbb{D} which fix the boundary pointwise and the set E_n^0 setwise. Equip this set of homeomorphisms with compact open topology. The connected components of this space form a group, called mapping class group of the n -punctured disc. This group is isomorphic to \mathcal{B}_n (see [9] or the recollection in chapter 2). Denote by \mathfrak{m}_b the connected component which corresponds to the braid b .

For a homeomorphism φ of a compact topological space its topological entropy $h(\varphi)$ is an invariant which measures the complexity of its behaviour in terms of iterations. It is defined in terms of the action of the homeomorphism on open covers of the compact space. A precise definition of the topological entropy is recalled in chapter 3 (see also the papers [1] or [12]).

For a braid b we define its entropy as

$$(1.1) \quad h(b) = \inf\{h(\varphi) : \varphi \in \mathfrak{m}_b\}.$$

The value is invariant under conjugation with self-homeomorphisms of the closed disc \mathbb{D} , hence it does not depend on the position of the set of punctures and on the choice of the representative of the conjugacy class \hat{b} . We write $h(\hat{b}) = h(b)$.

The entropy of surface homeomorphisms was considered first in a paper of Fathi and Shub in [12]. The *Astérisque* volume [12] is devoted to Thurston's work on surface homeomorphisms. Thurston considered connected closed Riemann surfaces X and self-homeomorphisms φ of X . His interest in this topic was motivated by the geometrization conjecture. He considered a class of homeomorphisms, which he called pseudo-Anosov, and proved that the mapping torus of any pseudo-Anosov mapping admits a complete hyperbolic metric of finite volume. This was one of the eight geometries. Thurston used dynamical methods (Markov partitions). Moreover, Thurston obtained a classification of mapping classes of self-homeomorphisms of closed surfaces into reducible and irreducible mapping classes and proved that each irreducible mapping class either contains a pseudo-Anosov homeomorphism or a periodic mapping. (The reader who is not familiar with these notions may consult chapter 2.) Detailed proofs of Thurston's theorems are given in [12].

Fathi and Shub considered in their paper the entropy of self-homeomorphisms of closed Riemann surfaces of genus at least 2. They showed that pseudo-Anosov self-homeomorphisms of such Riemann surfaces are entropy minimizing in their mapping class. (See [12] or chapters 2 and 3 for precise definitions and statements.) Moreover, Fathi and Shub related the entropy to Teichmüller's theorem and quasiconformal dilatation. The study can be extended to Riemann surfaces with punctures. This seems to be folklore among specialists. Being unable to find a reference, we include the respective statement and a short proof in chapter 3 (see below Theorem 3.2).

The case of braids concerns mapping classes of Riemann surfaces of second kind. Notice that also the entropy of reducible self-homeomorphisms of closed surfaces inevitably leads to the case of mapping classes of Riemann surfaces of second kind. We did not find a rigorous treatment in the literature and provide such a treatment in chapter 3. The key result in this respect is the Theorem 4 which will be formulated in the end of the introduction and will be proved in chapter 3 below. It describes the relation of the entropy of irreducible mapping classes on punctured Riemann surfaces and the entropy of associated relative mapping classes on second kind Riemann surfaces (i.e. mapping classes fixing the boundary circles pointwise). Theorem 4 reduces the computation of the entropy of irreducible braids (1.1) (i.e. of irreducible mapping classes \mathfrak{m}_b) to the computation of the entropy of the associated mapping classes $\mathfrak{m}_{b,\infty}$ on the $(n+1)$ -punctured Riemann sphere.

The entropy of reducible mapping classes is studied in chapter 7. We did not find a rigorous treatment in the literature. The main theorem in this respect is Theorem 7.1. In the end of the introduction we state a version of Theorem 7.1 which is shorter and easier to state than Theorem 7.1 but contains less information.

The entropy of surface homeomorphisms has received a lot of attention and has been studied intensively. For instance, it is known that the entropy of any irreducible braid is the logarithm of an algebraic number. Further, the lowest non-vanishing entropy h_n among irreducible braids on n strands, $n \geq 3$, has been

estimated from below by $\frac{\log 2}{4}n^{-1}$ ([30]) and has been computed for small n . There is an algorithm for computing the entropy of irreducible braids (respectively, of irreducible mapping classes) ([8]). Fluid mechanics related to stirring devices uses the entropy of the arising braids as a measure of complexity.

It turns out that the dynamical aspect and the conformal aspect are related. The following theorem holds.

Theorem 1. *For each conjugacy class of braids $\hat{b} \in \hat{\mathcal{B}}_n$, $n \geq 2$, the following equality holds*

$$\mathcal{M}(\hat{b}) = \frac{\pi}{2} \frac{1}{h(\hat{b})}.$$

The proof of the theorem is given first in the irreducible case. The upper bound for the conformal module of irreducible braids is proved in chapter 4. The proof uses Teichmüller theory, its relation to braids, and some less commonly known preparatory facts, included into chapter 2, as well as the theorems on entropy of chapter 3 (in particular, Theorem 3.3). A key role is played by Royden's Theorem on equality of the Teichmüller metric and the Kobayashi metric on Teichmüller space [32].

Specialists may expect to find a relation to the following known fact in Teichmüller theory.

Let X be a closed connected Riemann surface or a finite open connected Riemann surface of first kind. Suppose φ is a pseudo-Anosov self-homeomorphism of X , so its induced modular transformation φ^* on the Teichmüller space $\mathcal{T}(X)$ of X is hyperbolic. Take the quotient of $\mathcal{T}(X)$ by the action of the group generated by φ^* . Consider the largest conformal module of an annulus which admits a holomorphic mapping into the quotient such that the monodromy is φ^* . Then the product of this conformal module with the entropy $h(\varphi)$ equals $\frac{\pi}{2}$.

The space \mathfrak{P}_n is not a Teichmüller space. However, it is related to the Teichmüller space of the $(n+1)$ -punctured sphere as follows. The configuration space $C_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n, z_i \neq z_j \text{ for } i \neq j\}$ is a holomorphic covering of \mathfrak{P}_n . Consider the quotient $C_n(\mathbb{C})/\mathcal{A}$ of the configuration space by the diagonal action of the group \mathcal{A} of Möbius transformations that fix infinity. The Teichmüller space of the $(n+1)$ -punctured sphere is the universal covering space of $C_n(\mathbb{C})/\mathcal{A}$. The covering projection is holomorphic. We use this fact in chapter 4 to associate to a holomorphic map f of an annulus into \mathfrak{P}_n a modular transformation of the Teichmüller space. Analysing the relation between this modular transformation and the conjugacy class of braids represented by f , and applying Royden's Theorem, a Theorem of Bers (see also Corollary 2.1 of the present paper) and Theorem 3.2, one obtains the upper bound of the conformal module in the irreducible case.

The lower bound for the conformal module of irreducible braids is proved in chapter 5. To obtain from a hyperbolic modular transformation a holomorphic mapping of an annulus into \mathfrak{P}_n one needs to associate to a holomorphic mapping of a disc into the quotient $C_n(\mathbb{C})/\mathcal{A}$ a suitable holomorphic section to configuration space. Its existence is based on the fact that an annulus A has trivial second cohomology $H^2(A, \mathbb{Z})$.

The proof of Theorem 1 in the reducible case is given in chapter 7. It uses chapter 6 where we describe the decomposition of braids (considered as isotopy classes of geometric braids) into irreducible components, and Theorem 7.1 on the

entropy of reducible homeomorphisms. Theorem 7.1 is of independent interest. It expresses the entropy of a reducible mapping class by the maximum of the entropies of its irreducible components. For the definition of irreducible components of a mapping class see chapter 2.

There are several comments. It is known that an irreducible conjugacy class of braids has vanishing entropy iff it is represented by a periodic braid. In the same way it has infinite conformal module iff it is represented by a periodic braid. Moreover, in this case the conjugacy class can be represented by a holomorphic mapping of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ into \mathfrak{P}_n . Hence, all braids on two strands have vanishing entropy and their conjugacy class can be represented by a holomorphic mapping of \mathbb{C}^* into \mathfrak{P}_2 .

The conjugacy class of a reducible braid has vanishing entropy (equivalently, it has infinite conformal module) iff all irreducible components can be represented by periodic braids. In this case the conjugacy class can be represented by a holomorphic mapping of any annulus of finite conformal module into \mathfrak{P}_n but not by a holomorphic mapping of the punctured plane into \mathfrak{P}_n (Proposition 8.1). Thus, the conformal point of view (representing a braid by a holomorphic mapping of an annulus into \mathfrak{P}_n) gives more information than entropy does.

As a consequence we see that “most” conjugacy classes of braids $\hat{b} \in \hat{\mathcal{B}}_n$ for $n > 2$ have positive entropy, or, equivalently finite conformal module.

Theorem 1 has the following corollary.

Corollary 1. *For each $\hat{b} \in \hat{\mathcal{B}}_n$ ($n \geq 2$) and each nonzero integer l*

$$\mathcal{M}(\hat{b}^l) = \frac{1}{|l|} \mathcal{M}(\hat{b}).$$

Notice that the inequality $\mathcal{M}(\hat{b}^l) \geq \frac{1}{|l|} \mathcal{M}(\hat{b})$ can be proved directly. Indeed, take an annulus A which admits a holomorphic map f into \mathfrak{P}_n that represents \hat{b} and has conformal module close to that of \hat{b} . The lift of f to an unramified $|l|$ -covering of A represents \hat{b}^l . The estimate follows.

The corollary shows that for $n > 2$ for each positive number ε there are conjugacy classes of braids with conformal module less than ε .

The concept of the conformal module of conjugacy classes of braids can be applied to the following problem of algebraic geometry.

Problem. *Let X be a connected open Riemann surface and let $f : X \rightarrow \mathfrak{P}_n$ be a separable quasipolynomial of degree n on X . Is f isotopic to a holomorphic quasipolynomial on X ?*

In general the answer is negative and Lemma 1 provides obstructions. Runge’s approximation theorem implies the following lemma.

Lemma 2. *In the situation of the Problem there exists a domain $\tilde{X} \subset X$ which is diffeomorphic to X such that the restriction $f|_{\tilde{X}}$ is isotopic to a holomorphic quasipolynomial on \tilde{X} .*

Indeed, let Q be a 1-skeleton for X (a bouquet of circles which is a strong deformation retract of X). Consider the restriction $f|_Q$ and write it as $f(z, \zeta) = \zeta^n + a_{n-1}(z)\zeta^{n-1} + \dots + a_0(z)$, $z \in Q$, $\zeta \in \mathbb{C}$. The continuous functions $a_j(z)$, $z \in Q$, can be approximated uniformly on Q by meromorphic functions on X . See [24] for Mergelyan type approximation (approximation of continuous functions on Q by holomorphic functions in a neighbourhood of Q) and [33] for Runge approximation by meromorphic functions on closed Riemann surfaces (here, approximation of holomorphic functions in a neighbourhood of Q by holomorphic functions on X). We obtain approximating holomorphic quasipolynomials on X which are not necessarily separable. But on Q they are close to $f|_Q$, hence their restrictions to Q are separable and isotopic to $f|_Q$. Hence an approximating holomorphic quasipolynomial is isotopic to f on a small enough neighbourhood $\tilde{X} \subset X$ of Q with \tilde{X} diffeomorphic to X (and in fact, it is isotopic to f on any open subset U of X such that U retracts to Q and f is separable on U).

Lemma 2 can be reformulated using the following terminology.

Definition 2. *Let X and Y be open Riemann surfaces and let $f : X \rightarrow \mathfrak{P}_n$ be a separable quasipolynomial on X . Let $w : X \rightarrow Y$ be a homeomorphism. The homeomorphism w can be interpreted as a new conformal structure on X . We say that f is isotopic to an algebroid function for the conformal structure w if $f \circ w^{-1}$ is isotopic to an algebroid function on Y .*

Lemma 2 states that for each separable quasipolynomial on a connected open Riemann surface X there exists a conformal structure on X for which the quasipolynomial is isotopic to a holomorphic quasipolynomial.

In the following theorem we consider quasipolynomials of degree 3. The theorem shows that the obstructions for a separable quasipolynomial to be isotopic to a holomorphic quasipolynomial are discrete.

Theorem 2. *Let X be a torus with a disc removed. There are four conformal structures w_j , $j = 1, \dots, 4$, of second kind on X such that the following holds for each separable quasipolynomial of degree 3 which is isotopic to an algebroid function for each w_j , $j = 1, \dots, 4$.*

- (1) *The quasipolynomial is isotopic to an algebroid function for each conformal structure of second kind on X .*
- (2) *The quasipolynomial extends to a (smooth) separable quasipolynomial on the closed torus.*
- (3) *If the quasipolynomial is irreducible then it is isotopic to an algebroid function also for each conformal structure of first kind on X (i.e. on punctured tori). Moreover, the zero set of the algebroid function on a punctured torus extends to a saturated set (i.e. to a union of leaves) of a non-singular holomorphic foliation on the total space of a holomorphic line bundle on the closed torus.*
- (4) *A reducible separable quasipolynomial on a closed torus cannot be isotopic to a holomorphic quasipolynomial on the punctured torus unless it is isotrivial.*

A quasipolynomial of degree n on a closed torus X^c is called isotrivial if in certain smooth coordinates $(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ on X^c the quasipolynomial is

isotopic to the following:

$$f(\zeta, (x, y)) = \prod_{j=1}^n \left(\zeta - e^{\frac{2\pi i j}{n} x} \right), \quad \zeta \in \mathbb{C}, \quad (x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}.$$

The choices of the conformal structures in Theorem 2 can be made explicit. (There are many choices satisfying the conditions.) Each of the four conformal structures w_j corresponds to an element e_j of the fundamental group $\pi_1(X, x_0)$. (There is also some freedom for the choice of the four elements of $\pi_1(X, x_0)$.) The only requirements are the following. The Riemann surface $w_j(X)$ contains an annulus representing e_j and of conformal module strictly bigger than $\frac{\pi}{2} \left(\log \frac{3+\sqrt{5}}{2} \right)^{-1}$. Moreover, one of the $w_j(X)$ contains an annulus of conformal module bigger than $\frac{\pi}{2} \left(\log \frac{3+\sqrt{5}}{2} \right)^{-1}$ for which one the boundary components is equal to ∂X . Note that $\log \frac{3+\sqrt{5}}{2}$ is the smallest non-vanishing entropy among 3-braids (See e.g. [34]).

The following lemma demonstrates the influence of holomorphicity of a quasipolynomial on a thick annulus adjacent to the boundary ∂X .

Lemma 3. *Let X be a closed Riemann surface of positive genus with a geometric disc removed. Suppose f is an irreducible separable algebroid function of degree 3 on X . Suppose X contains a domain A , one of whose boundary components coincides with the boundary circle of X , such that A is conformally equivalent to an annulus of conformal module strictly larger than $\frac{\pi}{2} \left(\log \frac{3+\sqrt{5}}{2} \right)^{-1}$. Then f is solvable over A .*

The outline of the simple argument is the following. The estimate of the conformal module of A and the properties of the covering $\mathfrak{S}_f \rightarrow X$ allow only the solvable case or the case of periodic conjugacy classes $\hat{b}_{f,A}$ which correspond to conjugacy classes of 3-cycles. The latter is impossible for conjugacy classes of products of braid commutators. For details see chapter 8.

The proof of Theorem 2 does not use Theorem 1. However, the precise estimate of the conformal module of the annuli, which are required to be contained in the $w_j(X)$, is based on Theorem 1 and on known results on the entropy of braids on three strands.

Another application of the concept of the conformal module of braids concerns the result of Gorin and Lin [15], one of the first results which makes explicit use of the concept of conformal module. The result of [15] was extended by Zjuzin [41] and Petunin.

Theorem 1 and the estimate of entropy [30] allow to re-interpret the results of [15] and [41] and to give a new conceptional proof of a slightly stronger result. For details see chapter 8.

There is also the concept of the conformal module of braids rather than of conjugacy classes of braids. This notion is based on the conformal module of rectangles which admit holomorphic mappings into \mathfrak{P}_n with suitable boundary conditions on a pair of opposite sides. Recall that the conformal module of a rectangle with sides parallel to the coordinate axes is the ratio of the side lengths of horizontal and vertical sides. The conformal module of braids is a finer invariant than entropy. If suitably defined it is more appropriate for application, in particular, to real algebraic geometry. In the case of three-braids there are two versions differing by the

boundary conditions on horizontal sides of rectangles. (Both should be used.) For the first version one requires that horizontal sides are mapped to polynomials with all zeros on a real line. In the second version one requires that two of the zeros have equal distance from the third. These are the cases appearing on the real axis for polynomials with real coefficients. The invariant can be studied by quasiconformal mappings and elliptic functions. The situation for braids on more than 3 strands is more subtle. We intend to come back to this concept in a later paper.

The following theorem gives an application of the concept of the conformal module to elliptic fiber bundles.

A smooth (respectively, holomorphic) elliptic fiber bundle over a Riemann surface X is a triple $\mathfrak{F} = (\mathcal{X}, \mathcal{P}, X)$, where \mathcal{X} is a smooth 4-manifold (respectively, a complex surface), \mathcal{P} is a smooth (respectively, holomorphic) proper surjection $\mathcal{P} : \mathcal{X} \rightarrow X$, and each fiber $\mathcal{P}^{-1}(x)$, $x \in X$, is a closed torus.

Theorem 3. *Let S be a smooth torus with a hole. There are four conformal structures $w_j : S \rightarrow X_j$ of second kind on S such that the following statements hold.*

If a smooth elliptic fiber bundle \mathfrak{F} on S is isotopic to a holomorphic fiber bundle for each of the four structures then it is isotopic to a holomorphic bundle for each conformal structure of second kind on S . Moreover, the smooth bundle on S extends to a smooth bundle on a smooth closed torus S^c containing S .

The bundle is of one of the following special kinds.

- 1) *The extended bundle on S^c is isotopic to a holomorphic bundle for each conformal structure on S^c , equivalently, the bundle is isotrivial (i.e. a finite covering of the bundle is the trivial bundle).*
- 2) *For an arbitrary conformal structure $w : S \rightarrow X$ of second kind the push-forward of the bundle to X is isotopic to a quotient of a holomorphic fiber bundle on X with fiber $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Unless the bundle is isotrivial it is not isotopic to a holomorphic bundle for any conformal structure of first kind.*

The second option means the following. A holomorphic fiber bundle on X with fiber \mathbb{C}^* is a triple $(\mathcal{X}^*, \mathcal{P}^*, X)$, where \mathcal{X}^* is a complex surface and \mathcal{P}^* is a proper holomorphic submersion such that the fibers $(\mathcal{P}^*)^{-1}(x)$ are equal to \mathbb{C}^* for all $x \in X$.

By a quotient of a bundle we mean the following. Let G be a group of biholomorphic self-maps of \mathcal{X}^* which preserve all fibers. Require that G acts freely and properly discontinuously. The holomorphic surjection \mathcal{P}^* descends to a proper holomorphic surjection $\mathcal{P}_G^* : \mathcal{X}^*/G \rightarrow X$. The bundle $(\mathcal{X}^*/G, \mathcal{P}_G^*, X)$ is the quotient bundle.

Automatically the group G is generated by a single element which acts in each fiber by multiplication with a complex number of absolute value different from one.

We state now the theorems on the entropy of mapping classes of Riemann surfaces of second kind and on the entropy of reducible mapping classes.

Theorem 4 (The entropy of mapping classes of Riemann surfaces of second kind). *Let X be a connected closed Riemann surface with a set E of distinguished points. Assume that $X \setminus E$ is hyperbolic (i.e. covered by \mathbb{C}_+). Let φ be a pseudo-Anosov*

self-homeomorphism of X which fixes E setwise. Suppose $z_0 \in E$ is a fixed point of φ , $\varphi(z_0) = z_0$, and the entropy $h(\varphi)$ of φ is finite.

Then φ is isotopic through self-homeomorphisms of X which fix E to a homeomorphism φ_0 which fixes a topological disc around z_0 pointwise and has the same entropy $h(\varphi_0) = h(\varphi)$ as φ .

Notice that the complement X_0 of X of an open disc around z_0 is a bordered Riemann surface (the closure of a Riemann surface of second kind), and the restriction $\varphi_0|X_0$ is a self-homeomorphism of X_0 with entropy $h(\varphi_0|X_0) = h(\varphi)$.

Consider now a reducible braid $b \in \mathcal{B}_n$. There exists an admissible system \mathcal{C} of simple closed Jordan curves in $\mathbb{C} \setminus E_n^0$ and a representative $\varphi \in \mathfrak{m}_{b,\infty}$ which fixes the complement $(\mathbb{C} \setminus E_n^0) \setminus \bigcup_{C \in \mathcal{C}} C$ setwise. (For the definition of an admissible system of curves see e.g. chapter 2.)

Following Bers we associate to the braid b and the system \mathcal{C} a nodal Riemann surface Y and a mapping class on Y . The nodal Riemann surface Y is the image of X by a continuous mapping w which collapses each curve C_j to a point and is a homeomorphism on $X \setminus \bigcup_{C \in \mathcal{C}} C$. The images of the curves are called nodes. We

consider the mapping class $\mathring{\mathfrak{m}}_{b,\infty}$ on Y which is induced by $\mathfrak{m}_{b,\infty}$ and w . Identify mapping classes on Y with mapping classes on the closure \overline{Y} with distinguished points being the punctures and the nodes. By an abuse of notation we denote the class on the closure \overline{Y} obtained by this identification also by $\mathring{\mathfrak{m}}_{b,\infty}$. (For definitions see also chapter 2, for more details see chapters 6 and 7.) The following theorem holds.

Theorem 5. $h(\mathfrak{m}_b) = h(\mathring{\mathfrak{m}}_{b,\infty})$.

The images under w of the connected components of $(\mathbb{C} \setminus E_n^0) \setminus \bigcup_{C \in \mathcal{C}} C$ are called the parts of Y . Notice that the parts of the nodal surface Y are punctured Riemann spheres (i.e. each part equals the Riemann sphere with the nodes and some distinguished points removed.) The mappings of the mapping class permute the parts in cycles. For each part Y_j there is a natural number N_j such that the N_j th iterate of the restrictions $\mathring{\varphi}^{N_j}|Y_j$ of the mappings $\mathring{\varphi}$ of the class $\mathring{\mathfrak{m}}_{b,\infty}$ to the part Y_j fix the part. Moreover, the mapping class $\mathring{\varphi}^{N_j}|Y_j$ contains either a periodic map or a pseudo-Anosov map. Thus Theorem 5 allows to reduce the computation of the entropy of reducible braids to the computation of the entropy of irreducible mapping classes. For details see chapter 7, Theorem 7.1.

Part of the results of the present paper were announced without proof in [18].

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CHAPTER 2

Some recollection: Braids and Teichmüller Theory

The proof of the main Theorem 1 makes essential use of the theory of braids, mapping classes and Teichmüller theory. For completeness and convenience of the reader and for fixing notation and terminology we give a recollection of basic facts used for the proof. Readers who know the material may skip this chapter and consult it later for notation and for some less standard facts which we include in this chapter.

Braids. (For details see [9], [22].) Most intuitively, braids are described in terms of geometric braids. We will use here the complex plane \mathbb{C} though the complex structure will play a role only later. Points $(z_1, \dots, z_n) \in \mathbb{C}^n$ we consider as ordered tuples of points in \mathbb{C} . Usually the points $z_j \in \mathbb{C}$ will be required to be pairwise distinct, in other words, we require that (z_1, \dots, z_n) belongs to the *configuration space* $C_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j\}$. Denote by \mathcal{S}_n the symmetric group. Each permutation in \mathcal{S}_n acts on $C_n(\mathbb{C})$ by permuting the coordinates. The quotient $C_n(\mathbb{C})/\mathcal{S}_n$ is called the symmetrized configuration space. Its elements we denote by $\{z_1, \dots, z_n\}$ and consider them as unordered n -tuples of points in \mathbb{C} or as subset E of \mathbb{C} consisting of exactly n points. For a subset A of \mathbb{C} we will put $C_n(A) = C_n(\mathbb{C}) \cap A^n$.

The configuration space inherits the topology and complex structure from \mathbb{C}^n . The symmetrized configuration space is given the quotient topology and quotient complex structure. Note that \mathcal{S}_n acts freely and properly discontinuously on $C_n(\mathbb{C})$. The canonical projection $\mathcal{P}_{\text{sym}} : C_n(\mathbb{C}) \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$ is holomorphic.

Let $E_n \in C_n(\mathbb{C})/\mathcal{S}_n$ be a subset of \mathbb{C} containing exactly n points (equivalently, E_n is an unordered n -tuple of points in \mathbb{C}). A geometric braid in $[0, 1] \times \mathbb{C}$ with base point E_n is a collection of n mutually disjoint arcs in the cylinder $[0, 1] \times \mathbb{C}$ which joins the set $\{1\} \times E_n$ in the top with the point $\{0\} \times E_n$ in the bottom of the cylinder and intersects each fiber $\{t\} \times \mathbb{C}$ along an unordered n -tuple $E_n(t)$ of points. The arcs are called the strands of the geometric braid. In other words, a geometric braid with base point E is a continuous mapping $[0, 1] \ni t \xrightarrow{f} C_n(\mathbb{C})/\mathcal{S}_n$ with $f(0) = f(1) = E_n$. Two geometric braids with base point E_n are called isotopic if there is a continuous family of geometric braids with base point E_n joining them. A braid on n strands with base point E_n is an isotopy class of geometric braids with base point E_n . Isotopy classes of geometric braids with base point E_n form a group. The operation is obtained by putting one geometric braid on the bottom of another.

Take a geometric braid with base point E_n . Assigning to each point in $\{1\} \times E_n$ the point in $\{0\} \times E_n$ which belongs to the same strand we obtain a permutation of

E_n . It depends only on the isotopy class which we denote by b . This permutation is denoted by $\tau_n(b)$. The braid b is called pure if $\tau_n(b)$ is the identity.

Algebraically, n -braids are represented as elements of the Artin group \mathcal{B}_n . This is the group with generators denoted by $\sigma_1, \dots, \sigma_{n-1}$ and relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$, and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, \dots, n - 2$. An isomorphism between \mathcal{B}_n and isotopy classes of geometric braids with given base point can be obtained as follows. Choose a projection Pr of \mathbb{C} onto the real line \mathbb{R} which is injective on E_n . The projection $[0, 1] \times \mathbb{C} \ni (t, z) \rightarrow (t, \text{Pr } z) \in [0, 1] \times \mathbb{R}$ assigns to each geometric braid a braid diagram (at intersection points of images of strands it is indicated which strands is “over” and which strand in “under”). The generator σ_i corresponds to a positive half-twist of the i -th and the $(i + 1)$ -st strand. Isotopy classes of geometric braids correspond to equivalence classes of diagrams. The latter can be interpreted as elements of the Artin group. The isomorphism from the group of isotopy classes of geometric braids to the Artin group depends on the base point E_n and the projection Pr . Different isomorphisms are related by conjugation with an element of the Artin group.

Arnol'd interpreted the symmetrized configuration space $C_n(\mathbb{C})/\mathcal{S}_n$ as the space of monic polynomials \mathfrak{P}_n of degree n without multiple zeros. Denote by $\overline{\mathfrak{P}}_n$ the set of *all* monic polynomials of degree n . If we assign to each unordered n -tuple $E_n = \{z_1, \dots, z_n\}$ (this time the z_j are not necessarily pairwise distinct) the monic polynomial $\prod_{j=1}^n (z - z_j) = a_0 + a_1 z + \dots + z^n$ whose set of roots equal E_n , we obtain a bijection onto the set $\overline{\mathfrak{P}}_n$. The set of monic polynomials of degree n can also be parametrized by the ordered n -tuple $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{C}^n$ of the coefficients. Hence, we obtain a bijection between $C_n(\mathbb{C})/\mathcal{S}_n$ and the set of points $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{C}^n$ which are coefficients of polynomials without multiple zeros. There is a polynomial D_n in the variables a_0, \dots, a_{n-1} , called the discriminant, which vanishes exactly if the polynomial with these coefficients has multiple zeros. Hence, the symmetrized configuration space $C_n(\mathbb{C})/\mathcal{S}_n$ can be identified with $\mathbb{C}^n \setminus V_{D_n}$ where $V_{D_n} \stackrel{\text{def}}{=} \{(a_0, \dots, a_{n-1}) \in \mathbb{C}^n : D_n(a_0, \dots, a_{n-1}) = 0\}$. The identification is actually a biholomorphic map. Thus, $C_n(\mathbb{C})/\mathcal{S}_n$ is biholomorphic to a pseudoconvex domain in \mathbb{C}^n , even stronger, it is biholomorphic to the complement of a complex hypersurface in \mathbb{C}^n . The space $\mathfrak{P}_n \cong C_n(\mathbb{C})/\mathcal{S}_n$ received much attention in connection with problems of algebraic geometry. For instance, motivated by his interest in the Thirteen's Hilbert problem, Arnol'd [4] computed its topological invariants.

Geometric braids with base point E_n were interpreted as pathes in $C_n(\mathbb{C})/\mathcal{S}_n \cong \mathfrak{P}_n$ with initial and terminating point equal to E_n , in other words, as loops in this space with base point E_n . Isotopy classes of geometric braids with base point E_n correspond to homotopy classes of loops in \mathfrak{P}_n with base point E_n , in other words, to elements of the fundamental group $\pi_1(\mathfrak{P}_n, E_n)$ of \mathfrak{P}_n with base point E_n . Thus \mathcal{B}_n is isomorphic to $\pi_1(\mathfrak{P}_n, E_n)$. A change of the base point leads to an automorphism of \mathcal{B}_n defined by conjugation with an element of \mathcal{B}_n . Denote by $\hat{\mathcal{B}}_n$ the set of conjugacy classes of \mathcal{B}_n . Its elements $\hat{b} \in \hat{\mathcal{B}}_n$ can be interpreted as free homotopy classes of loops in \mathfrak{P}_n . In other words, two geometric braids $f_0 : [0, 1] \rightarrow \mathfrak{P}_n$, $f_1 : [0, 1] \rightarrow \mathfrak{P}_n$, represent the same class \hat{b} , if there is a free

homotopy joining them, i.e. there exists a continuous mapping

$$h : [0, 1] \times [0, 1] \rightarrow \mathfrak{P}_n$$

such that $h(t, 0) = f_0(t)$, $h(t, 1) = f_1(t)$, for $t \in [0, 1]$, and $h(s, 0) = h(s, 1)$ for $s \in [0, 1]$. We may consider the continuous family of braids $f_s : [0, 1] \rightarrow \mathfrak{P}_n$ with variable base point as a free isotopy of braids.

We will also use the following terminology. A loop in \mathfrak{P}_n (i.e. a continuous map of the circle into \mathfrak{P}_n) is called a closed geometric braid. A free homotopy class of loops in \mathfrak{P}_n is called a closed braid. Closed braids correspond to elements of $\hat{\mathcal{B}}_n$.

Quasipolynomials and coverings. Consider quasipolynomials f of degree n on a topological space X , in other words, consider continuous mappings from X into $\overline{\mathfrak{P}}_n$. The topological properties of quasipolynomials are studied by their restriction to the complement $X \setminus D_f$ of their discriminant set D_f (the set which is mapped by f to polynomials of degree n with multiple zeros). In particular, the topological properties are determined by the restriction of f to a system of loops in $X \setminus D_f$ which represents a system of generators of the fundamental group $\pi_1(X \setminus D_f, x_0)$ with a base point $x_0 \in X \setminus D_f$. Notice that the restrictions of f to these loops defines loops in \mathfrak{P}_n .

A quasipolynomial of degree n induces a branched (topological) n -covering of X in the following way. Let $\mathfrak{S}_f \subset X \times \mathbb{C}$ be the zero set of f considered as a function of two variables. Let $\mathcal{P}_X : X \times \mathbb{C} \rightarrow X$ be the canonical projection. The mapping

$$p \stackrel{\text{def}}{=} \mathcal{P}_X | \mathfrak{S}_f : \mathfrak{S}_f \rightarrow X$$

is a branched topological n -covering of X . We say that the quasipolynomial lifts the covering. If the quasipolynomial is separable the n -covering is unramified.

If X is a Riemann surface the set \mathfrak{S}_f will always be given the structure of a Riemann surface in such a way that p is holomorphic. The quasipolynomial f is not required to be holomorphic.

Propositions 2.1 and 2.2 and Corollary 2.1 below summarize known results on quasipolynomials and coverings. (For details see e.g. [35], [16].) To formulate them we will use the following terminology. Two unramified holomorphic n -coverings $p : Y \rightarrow X$ and $\tilde{p} : \tilde{Y} \rightarrow X$ of a connected Riemann surface X are called equivalent if there is a conformal mapping \mathfrak{c} from Y onto \tilde{Y} such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\mathfrak{c}} & \tilde{Y} \\ & \searrow p & \swarrow \tilde{p} \\ & X & \end{array}$$

For a group G two homomorphisms $\Phi_1 : G \rightarrow \mathcal{S}_n$ and $\Phi_2 : G \rightarrow \mathcal{S}_n$ are called conjugate if there is an element $s \in \mathcal{S}_n$ such that

$$\Phi_2(a) = s^{-1} \Phi_1(a) s \quad \text{for each } a \in G.$$

PROPOSITION 2.1. *Let X be a connected closed Riemann surface or a connected bordered Riemann surface. Let $\pi_1(X, x_0)$ be the fundamental group of X with a given base point x_0 . The following statements hold.*

- (1) *There is a one-to-one correspondence between unramified holomorphic n -coverings $p : Y \rightarrow X$ with given label of points in the fiber $p^{-1}(x_0)$ and homomorphisms $\Psi : \pi_1(X, x_0) \rightarrow \mathcal{S}_n$.*
- (2) *There is a one-to-one correspondence between equivalence classes of unramified holomorphic n -coverings of X and conjugacy classes of homomorphisms from $\pi_1(X)$ into \mathcal{S}_n .*
- (3) *The connected components of the covering space Y of an unramified holomorphic covering $p : Y \rightarrow X$ are in bijective correspondence to the orbits of $p_*(\pi_1(x, z_0)) \subset \mathcal{S}_n$ on the set with n points. In particular, Y is connected iff $p_*(\pi_1(X, x_0))$ acts transitively. (p_* is the homomorphism $\pi_1(X, x_0) \rightarrow \mathcal{S}_n$ corresponding to p .)*
- (4) *Suppose X is a bordered Riemann surface with connected boundary. Denote by $\{\partial X\}$ an element of the fundamental group $\pi_1(X, x_0)$ which represents the free homotopy class of ∂X . Then the connected components of the boundary ∂Y correspond to the orbits of the single even permutation $p_*(\{\partial X\})$. $p_*(\{\partial X\})$ is the product of g commutators in \mathcal{S}_n . Here g is the genus of X .*

PROPOSITION 2.2. *Let again X be a connected closed Riemann surface or a connected bordered Riemann surface. Let $\pi_1(X, x_0)$ be the fundamental group of X with base point x_0 . The following statements hold.*

- (1) *Let $E_n \in \mathfrak{P}_n$ be a base point. The homomorphisms $\Phi : \pi_1(X, x_0) \rightarrow \mathcal{B}_n$ are in one-to-one correspondence to the isotopy classes of separable quasipolynomials of degree n on X with value E_n at x_0 and given label of points in E_n .*
- (2) *The (free) isotopy classes of separable quasipolynomials of degree n on X are in one-to-one correspondence to conjugacy classes of homomorphisms from $\pi_1(X)$ into \mathcal{B}_n .*
- (3) *Let X be a bordered Riemann surface with connected boundary. The restriction to ∂X of each separable quasipolynomial of degree n on X is a closed geometric braid. Restrictions to ∂X of free isotopy classes of separable quasipolynomials are closed braids, i.e. conjugacy classes of braids on n strands. The conjugacy classes of n -braids obtained in this way are the conjugacy classes of products of g commutators in \mathcal{B}_n . As before, g is the genus of X .*

COROLLARY 2.1. *Consider an unramified holomorphic n -covering $p : Y \rightarrow X$ of a finite Riemann surface X , and the induced homomorphism $p_* : \pi_1(X, x_0) \rightarrow \mathcal{S}_n$. The set of isotopy classes of separable quasipolynomials of degree n which lift p are in one-to-one correspondence to the set of conjugacy classes of homomorphisms $\Phi : \pi_1(X, x_0) \rightarrow \mathcal{B}_n$ which lift p_* (i.e. $\tau_n \circ \Phi = p_*$ for the natural projection $\tau_n : \mathcal{B}_n \rightarrow \mathcal{S}_n$). In particular, each unramified holomorphic n -covering $p : Y \rightarrow X$ can be lifted to a separable quasipolynomial of degree n on X .*

Mapping class groups. The braid group \mathcal{B}_n is isomorphic to the mapping class group of the n -punctured disc. For more details we introduce the following notation. Let A be a topological space, not necessarily compact, but paracompact. Let A_1

and A_2 be disjoint closed subsets of A . Denote by $\text{Hom}(A; A_1, A_2)$ the set of self-homeomorphisms of A that fix A_1 pointwise and A_2 setwise. We will also write $\text{Hom}(A; A_1)$ for $\text{Hom}(A; A_1, \emptyset)$, but in case we do not require that some points are fixed we write $\text{Hom}(A; \emptyset, A_2)$. We will also write $\text{Hom}(A)$ for $\text{Hom}(A; \emptyset, \emptyset)$. Equip the set $\text{Hom}(A; A_1, A_2)$ with compact open topology.

Let A be an oriented manifold. By $\text{Hom}^+(A; A_1, A_2)$ we denote the set of orientation preserving self-homeomorphisms in $\text{Hom}(A; A_1, A_2)$. It forms a group with respect to composition. The set of connected components of $\text{Hom}^+(A; A_1, A_2)$ is denoted by $\mathfrak{M}(A; A_1, A_2)$.

Let \bar{D} be the closed unit disc in \mathbb{R}^2 with boundary ∂D and interior D . Later we want to consider the standard complex structure on D and will write \mathbb{D} for the unit disc in the complex plane \mathbb{C} . Let $E_n^0 = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ be the “standard” subset of D containing n points. (We also identify E_n^0 with the respective unordered n -tuple of points.) The set $\mathfrak{M}(\bar{D}; \partial D, E_n^0)$ is commonly known as the mapping class group of the n -punctured disc. Note that $\mathfrak{M}(\bar{D}; \partial D, E_n^0)$ is isomorphic to $\mathfrak{M}(\bar{D} \setminus E_n^0; \partial D)$ since each element of $\mathfrak{M}(\bar{D} \setminus E_n^0; \partial D)$ extends continuously to each point of E_n^0 (the “punctures”). The points of E_n^0 are also called “distinguished points”. The homeomorphisms in $\text{Hom}^+(\bar{D}; \partial D, E_n^0)$ are called homeomorphisms of \bar{D} with distinguished points E_n^0 or homeomorphisms of the n -punctured disc. The connected component $\text{Hom}^0(\bar{D}; \partial D, E_n^0)$ of $\text{Hom}^+(\bar{D}; \partial D, E_n^0)$ containing the identity consists of the self-homeomorphisms of \bar{D} which can be joined to the identity by a continuous family of homeomorphisms in $\text{Hom}^+(\bar{D}; \partial D, E_n^0)$. In other words, it consists of homeomorphisms in $\text{Hom}^+(\bar{D}; \partial D, E_n^0)$ which are isotopic to the identity through homeomorphisms fixing ∂D and E_n^0 pointwise. We have

$$(2.1) \quad \mathfrak{M}(\bar{D}; \partial D, E_n^0) = \text{Hom}^+(\bar{D}; \partial D, E_n^0) / \text{Hom}^0(\bar{D}; \partial D, E_n^0).$$

The respective mapping class group $\mathfrak{M}(\bar{D}; \partial D, E_n)$ can be defined for any unordered n -tuple E_n of points in D .

More generally, let X be a compact connected surface (with or without boundary) and let E_n be a subset of X containing exactly n points. $\mathfrak{M}(X; \partial X, E_n) \cong \mathfrak{M}(X \setminus E_n, \partial X)$ is called the mapping class of the n -punctured Riemann surface. If X is a closed surface this space equals $\mathfrak{M}(X; \emptyset, E_n) \cong \mathfrak{M}(X \setminus E_n)$. Let again X be the closed disc \bar{D} . For a homeomorphism $\varphi \in \text{Hom}^+(\bar{D}; \partial D, E_n)$ we consider its restriction $\varphi|_D \in \text{Hom}^+(D; \emptyset, E_n)$ and the corresponding mapping class in $\mathfrak{M}(D; \emptyset, E_n)$. This mapping class is denoted by $\mathfrak{m}_\varphi^{\text{free}}$ and is called the free isotopy class of φ . We also call the elements of $\mathfrak{M}(\bar{D}; \partial D, E_n)$ the relative mapping classes. The restriction map $\varphi \rightarrow \varphi|_D$ defines a surjective mapping from $\mathfrak{M}(\bar{D}; \partial D, E_n)$ to $\mathfrak{M}(D; \emptyset, E_n)$. Indeed, each element of $\mathfrak{M}(D; \emptyset, E_n)$ contains representatives which extend to the boundary ∂D as the identity mapping on ∂D . There is a short exact sequence

$$(2.2) \quad 0 \rightarrow \mathcal{K} \rightarrow \mathfrak{M}(\bar{D}; \partial D, E_n) \rightarrow \mathfrak{M}(D; \emptyset, E_n) \rightarrow 0.$$

An element $\mathfrak{m} \in \mathfrak{M}(\bar{D}; \partial D, E_n)$ is in the kernel \mathcal{K} iff each representative is isotopic to the identity through homeomorphisms of \bar{D} which fix E_n setwise.

For the following theorem see e.g. [19].

THEOREM 2.1. *The kernel \mathcal{K} is generated by a Dehn twist about a simple closed curve γ in D which is homologous to ∂D in $D \setminus E_n$.*

A Dehn twist about a simple closed curve γ in D is an orientation preserving self-homeomorphism of D which is the identity outside an annulus with "waist curve" γ (i.e. the annulus A is a neighbourhood of γ) and equals a power of a full twist on the annulus. A full twist of a round annulus $\{r < |z| < R\}$ is an orientation preserving self-homeomorphism which is the identity on the boundary and which maps the radius $[r, R]$ to a curve $\rho e^{i\theta(\rho)}$, $\rho \in [r, R]$ such that the curve $\rho \rightarrow e^{i\theta(\rho)}$, $\rho \in [r, R]$, is a closed curve of index 1 around zero.

Let D^{Cl} be the one-point compactification of D . It can be obtained from \bar{D} by the continuous map $\mathcal{P}_{\partial D} : \bar{D} \rightarrow D^{\text{Cl}}$ which is the identity on the open set D and collapses the boundary ∂D to a single point which we denote z_∞ . Each self-homeomorphism of D extends to a self-homeomorphism of D^{Cl} which fixes z_∞ . This gives a group isomorphism $\mathfrak{M}(D; \emptyset, E_n) \rightarrow \mathfrak{M}(D^{\text{Cl}}; z_\infty, E_n)$. Composing the restriction map $\mathfrak{M}(\bar{D}; \partial D, E_n) \rightarrow \mathfrak{M}(D; \emptyset, E_n)$ with the group isomorphism we obtain a surjective homomorphism

$$(2.3) \quad \mathcal{H}_{\partial D} : \mathfrak{M}(\bar{D}; \partial D, E_n) \rightarrow \mathfrak{M}(D^{\text{Cl}}; z_\infty, E_n)$$

with kernel generated by a Dehn twist about a simple closed curve in D that is homologous to ∂D in $D \setminus E_n$.

Let X be a smooth surface with boundary. Denote by $\partial_1, \dots, \partial_N$ the boundary components. For each j we may consider the map \mathcal{P}_{∂_j} which collapses the boundary component ∂_j to a single point z_∞^j . Denote the corresponding map on mapping class groups by \mathcal{H}_{∂_j} . Denote by $(\text{Int } X)^{\text{Cl}}$ the N -point compactification $\text{Int } X \cup \{z_\infty^1, \dots, z_\infty^N\}$ of the interior $\text{Int } X$ of X . The composition of all \mathcal{H}_{∂_j} (in any order) defines a surjective homomorphism $\mathcal{H}_{\partial X}$

$$(2.4) \quad \mathcal{H}_{\partial X} : \mathfrak{M}(X; \partial X, E_n) \rightarrow \mathfrak{M}((\text{Int } X)^{\text{Cl}}; \{z_\infty^1, \dots, z_\infty^N\}, E_n)$$

with kernel generated by the Dehn twists around simple closed curves in $\text{Int } X$ that are homologous in $\text{Int } X \setminus E_n$ to ∂_j . The map $\mathcal{H}_{\partial X}$ acts by conjugation on $\text{Int } X$, followed by extension across the points $\{z_\infty^1, \dots, z_\infty^N\}$.

We equip now the interior of the surfaces with complex structure. In other words, we consider a closed Riemann surface or a bordered Riemann surface. Recall that, following Ahlfors, a smooth compact surface with boundary whose interior is a Riemann surface is called a bordered Riemann surface. Consider first the closed unit disc $\bar{\mathbb{D}}$ in the complex plane \mathbb{C} . Let $E_n \subset \mathbb{D}$ be a set containing exactly n points. Choose a homeomorphism of \mathbb{D} onto \mathbb{C} which fixes pointwise a disc that contains E_n . Conjugation in $\text{Hom}^+(\mathbb{D}; \emptyset, E_n)$ with the inverse of this homeomorphism leads to an isomorphism

$$\text{Is}_\infty : \mathfrak{M}(\bar{\mathbb{D}}^{\text{Cl}}; z_\infty, E_n) \rightarrow \mathfrak{M}(\mathbb{P}^1; \infty, E_n).$$

The isomorphism Is_∞ acts by conjugation with a homeomorphism from $\bar{\mathbb{D}}^{\text{Cl}}$ to \mathbb{P}^1 . It does not depend on the choice of the conjugating homeomorphism, and is therefore canonical. We obtain a homomorphism $\mathcal{H}_\infty = \text{Is}_\infty \circ \mathcal{H}_{\partial \mathbb{D}}$,

$$(2.5) \quad \mathcal{H}_\infty : \mathfrak{M}(\bar{\mathbb{D}}; \partial \mathbb{D}, E_n) \rightarrow \mathfrak{M}(\mathbb{P}^1; \infty, E_n),$$

which can be explicitly described as follows. Take a mapping class $\mathbf{m} \in \mathfrak{M}(\bar{\mathbb{D}}; \partial \mathbb{D}, E_n)$. Represent it by a homeomorphism φ . Extend φ to a self-homeomorphism φ_∞ of \mathbb{P}^1 by putting $\varphi_\infty = \varphi$ on $\bar{\mathbb{D}}$ and $\varphi_\infty = \text{id}$ outside $\bar{\mathbb{D}}$. Let $\mathbf{m}_\infty \in \mathfrak{M}(\mathbb{P}^1; \infty, E_n)$

be the mapping class of φ_∞ . It depends only on the mapping class \mathbf{m} of φ in $\mathfrak{M}(\mathbb{D}; \partial\mathbb{D}, E_n)$, not on the choice of the representative φ . We have

$$(2.6) \quad \mathcal{H}_\infty(\mathbf{m}) = \mathbf{m}_\infty \in \mathfrak{M}(\mathbb{P}^1; \infty, E_n).$$

More generally, let X be a bordered Riemann surface. Then there is a compact Riemann surface X^c and a diffeomorphism of X onto the closure of a domain in X^c . The diffeomorphism is conformal on $\text{Int } X$. The domain in X^c is obtained by removing from X^c a finite number of geometric discs. A geometric disc is a topological disc which lifts to a round disc on the universal covering of X equipped with standard metric. Identify X with the closure of the domain on X^c . Let $\partial_1, \dots, \partial_N$ be the boundary components of X and let $\delta_1, \dots, \delta_N$ be the open discs on X^c bounded by the ∂_j . Let $E_n \subset \text{Int } X$ be a finite set. For each $j = 1, \dots, N$ we pick a point $\zeta_j \in \delta_j$. Consider a homeomorphism of $\text{Int } X$ onto $X^c \setminus \{\zeta_1, \dots, \zeta_N\}$ which is the identity on $X \setminus \bigcup_{j=1}^N \bar{A}_j$, where the $\bar{A}_j \subset X$ are pairwise disjoint closed annuli not intersecting E_n , such that ∂_j coincides with one of the boundary circles of \bar{A}_j . Conjugation with the inverse of such a homeomorphism leads to a canonical isomorphism

$$(2.7) \quad \text{Is}_\zeta : \mathfrak{M}((\text{Int } X)^{\text{Cl}}; \{z_\infty^1, \dots, z_\infty^N\}, E_n) \rightarrow \mathfrak{M}(X^c; \{\zeta_1, \dots, \zeta_N\}, E_n).$$

Here we put $\zeta = (\zeta_1, \dots, \zeta_N)$. The isomorphism Is_ζ acts by conjugation with a homeomorphism from $(\text{Int } X)^{\text{Cl}}$ to X^c . We obtain a homeomorphism $\mathcal{H}_\zeta = \text{Is}_\zeta \circ \mathcal{H}_\partial$,

$$(2.8) \quad \mathcal{H}_\zeta : \mathfrak{M}(X; \partial X, E_n) \rightarrow \mathfrak{M}(X^c; \{\zeta_1, \dots, \zeta_N\}, E_n)$$

which can be described as follows. Take $\mathbf{m} \in \mathfrak{M}(X; \partial X, E_n)$. Let $\varphi \in \text{Hom}^+(X; \partial X, E_n)$ be a representing homeomorphism. Let φ^c be the extension of φ to X^c which is the identity outside X . Then $\mathcal{H}_\zeta(\mathbf{m}) = \mathbf{m}_\zeta$ where \mathbf{m}_ζ is the class of φ^c in $\mathfrak{M}(X^c; \{\zeta_1, \dots, \zeta_N\}, E)$.

We will now describe the isomorphism between the braid group \mathcal{B}_n and the mapping class group of the n -punctured disc. We use complex notation.

For any subset E of the unit disc and any self-homeomorphism $\psi \in \text{Hom}^+(\mathbb{D}, \emptyset, \emptyset)$ we put $\text{ev}_E \psi = \psi(E)$. If $E_n^0 = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ (considered as unordered tuple of n points or as set) then

$$(2.9) \quad \text{ev}_{E_n^0} \psi = \left\{ \psi(0), \psi\left(\frac{1}{n}\right), \dots, \psi\left(\frac{n-1}{n}\right) \right\}.$$

Let

$$(2.10) \quad e_n(\psi) = \left(\psi(0), \dots, \psi\left(\frac{n-1}{n}\right) \right)$$

assign to ψ the ordered n -tuple of points in $C_n(\mathbb{C})$. Note that $\mathcal{P}_{\text{sym}} e_n(\psi) = \text{ev}_{E_n^0} \psi$. The isomorphism between the mapping class group $\mathfrak{M}(\mathbb{D}, \partial\mathbb{D}, E_n^0)$ and the group of isotopy classes of geometric braids with base point E_n^0 is obtained as follows.

Let $\varphi \in \text{Hom}^+(\mathbb{D}; \partial\mathbb{D}, E_n^0)$. Consider a path $\varphi_t \in \text{Hom}^+(\mathbb{D}, \partial\mathbb{D})$, $t \in [0, 1]$, which joins φ with the identity. In other words, φ_t is a continuous family of self-homeomorphisms of \mathbb{D} which fix the boundary $\partial\mathbb{D}$ pointwise, such that $\varphi_0 = \varphi$

and $\varphi_1 = \text{id}$. We do not require that φ_t maps E_n^0 to itself. By the Alexander-Tietze theorem such a family exists for each $\varphi \in \text{Hom}^+(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$. Consider the evaluation map

$$[0, 1] \ni t \rightarrow \text{ev}_{E_n^0} \varphi_t = \left\{ \varphi_t(0), \dots, \varphi_t\left(\frac{n-1}{n}\right) \right\} \in C_n(\mathbb{C})/\mathcal{S}_n.$$

This is a geometric braid in the cylinder $[0, 1] \times \mathbb{D}$ (i.e. $\text{ev}_{E_n^0} \varphi_t \in C_n(\mathbb{D})/\mathcal{S}_n$ for each t). Its base point is

$$E_n^0 = \varphi_0(E_n^0) = \varphi_1(E_n^0).$$

Notice that the isotopy class of the obtained geometric braid depends only on the class of φ in $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$. The obtained mapping from $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$ to the group of braids with base point E_n^0 , hence to \mathcal{B}_n , is a homomorphism. It is, in fact, an isomorphism. This is a consequence of Proposition 2.3 below.

Let E_n be an arbitrary unordered n -tuple of points in \mathbb{D} . A continuous family of homeomorphisms $\varphi_t \in \text{Hom}^+(\overline{\mathbb{D}}; \partial \mathbb{D})$, $t \in [0, 1]$, is called a parametrizing isotopy of a geometric braid $f : [0, 1] \rightarrow C_n(\mathbb{D})/\mathcal{S}_n$ with base point E_n if $\varphi_1 = \text{id}$ and $\text{ev}_{E_n} \varphi_t = f(t)$, $t \in [0, 1]$.

PROPOSITION 2.3. *Let $f : [0, 1] \rightarrow C_n(\mathbb{D})/\mathcal{S}_n$ be a geometric braid in $[0, 1] \times \overline{\mathbb{D}}$ with any base point $E_n \in C_n(\mathbb{D})/\mathcal{S}_n$. Then there exists a parametrizing isotopy φ_t for f . If f is smooth then the parametrizing isotopy can be chosen to be smooth. In other words, φ_t can be chosen so that the map $[0, 1] \times \mathbb{D} \ni (t, z) \rightarrow (t, \varphi_t(z)) \in [0, 1] \times \overline{\mathbb{D}}$ is a smooth diffeomorphism.*

For the continuous version see e.g. [22]. For convenience of the reader we give the short proof of the smooth version below.

By the proposition the inverse of the mapping $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0) \rightarrow \mathcal{B}_n$ is obtained as follows. Take a braid $b \in \mathcal{B}_n$ and choose a representing geometric braid in $[0, 1] \times \mathbb{D}$. Consider a parametrizing isotopy φ_t . Associate to b the mapping class of the homeomorphism φ_0 .

Explicitly, the inverse mapping assigns to each generator $\sigma_j \in \mathcal{B}_n$ the class of the following homeomorphism which is called a half-twist around the interval $[\frac{j-1}{n}, \frac{j}{n}]$. Take two open discs D_1 and D_2 centered at the midpoint of the segment $[\frac{j-1}{n}, \frac{j}{n}]$, such that $[\frac{j-1}{n}, \frac{j}{n}] \subset D_1$, $\bar{D}_1 \subset D_2$, \bar{D}_2 does not contain points of E^0 other than $\frac{j-1}{n}$ and $\frac{j}{n}$. Define φ_{σ_j} to be the identity on $\overline{\mathbb{D}} \setminus D_2$ and to be counterclockwise rotation by the angle π on \bar{D}_1 . Extend this mapping by a homeomorphism of $\bar{D}_2 \setminus D_1$ which changes the argument of each point by a non-negative value at most equal to π . We will denote the mapping $\mathcal{B}_n \rightarrow \mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$ by Θ_n .

In the same way as we denoted the set of conjugacy classes of n -braids by $\hat{\mathcal{B}}_n$ we will denote the set of conjugacy classes of the mapping class group $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$ by $\hat{\mathfrak{M}}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$. Notice that the elements of $\hat{\mathcal{B}}_n$ are in one-to-one correspondence with the elements of $\hat{\mathfrak{M}}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$. Also, for an arbitrary Riemann surface X and a finite subset E of X we denote the conjugacy classes of the mapping class group by $\hat{\mathfrak{M}}(X; \partial X, E)$, respectively, by $\mathfrak{M}(X; \emptyset, E)$.

Proof of Proposition 2.3. Lift the mapping f to a mapping $\tilde{f} : [0, 1] \rightarrow C_n(\mathbb{D})$. Denote the coordinate functions of \tilde{f} by $f_j : [0, 1] \rightarrow \mathbb{D}$. Choose $\delta > 0$ so that for

each t the discs $\mathcal{U}_j(t)$ of radius δ around the points $f_j(t)$, $j = 1, \dots, n$, are pairwise disjoint subsets of \mathbb{D} .

For $\zeta \in \mathcal{U}_j(1)$ we put $\tilde{\varphi}_t(\zeta) = \zeta - f_j(1) + f_j(t)$, $j = 1, \dots, n$, $t \in [0, 1]$. Then for each t the map $\tilde{\varphi}_t$ is a diffeomorphism of $\mathcal{U}_j(1)$ onto $\mathcal{U}_j(t)$, $j = 1, \dots, n$. Consider for each j the tube $\mathfrak{T}_j = \bigcup_{t \in [0, 1]} (t, \mathcal{U}_j(t))$. Define a vector field v on the union $\bigcup_j \mathfrak{T}_j$ of the tubes, so that the graphs $\bigcup_{t \in [0, 1]} (t, f_j(t))$ of the f_j are integral curves. We may take $v(t, \zeta) = (1, f'_j(t))$ for (t, ζ) with $t \in [0, 1]$, $\zeta \in \mathcal{U}_j(t)$. (For each t and ζ the right hand side is a point in $\mathbb{R} \times \mathbb{C}$ which is identified with a vector in \mathbb{R}^3 .) Put $\tilde{\varphi}(t, \zeta) = (t, \tilde{\varphi}_t(\zeta))$ for $(t, \zeta) \in [0, 1] \times \bigcup_j \mathcal{U}_j(1)$. For $(t, \zeta) \in [0, 1] \times \mathcal{U}_j(1)$ we obtain

$$(2.11) \quad \frac{\partial}{\partial t} \tilde{\varphi}(t, \zeta) = \left(1, \frac{\partial}{\partial t} \varphi_t(\zeta)\right) = (1, f'_j(t)) = v(t, \tilde{\varphi}_t(\zeta)) = v(\tilde{\varphi}(t, \zeta)).$$

We obtain that the graphs of the f_j are integral curves of v . (The initial point is $(1, f_j(1))$.)

Let χ be a smooth function on $\bigcup_{j=1}^n \mathcal{U}_j(1)$, $0 \leq \chi \leq 1$, $\chi = 1$ on $\bigcup_{j=1}^n \{|\zeta - f_j(1)| < \frac{\delta}{2}\}$ and $\chi = 0$ in a neighbourhood of the boundary $\bigcup_{j=1}^n \partial \mathcal{U}_j(1)$. Consider the vector field $v(t, \zeta) \cdot \chi((\tilde{\varphi}_t)^{-1}(\zeta))$ in the union $\bigcup_{j=1}^n \mathfrak{T}_j$ of the tubes. Note that $\chi((\tilde{\varphi}_t)^{-1}(\zeta)) = 0$ if $(\tilde{\varphi}_t)^{-1}(\zeta)$ is in a neighbourhood of $\bigcup_{j=1}^n \partial \mathcal{U}_j(1)$, i.e. if ζ is in a neighbourhood of $\bigcup_{j=1}^n \partial \mathcal{U}_j(t)$. In the same way $\chi((\tilde{\varphi}_t)^{-1}(\zeta)) = 1$ if ζ is in a $\frac{\delta}{2}$ -neighbourhood of any of the $f_j(t)$. Extend the vector field by zero to the whole cylinder $[0, 1] \times \overline{\mathbb{D}}$. Denote the extended vector field by $V(t, \zeta)$. Let $X_\zeta(t)$, $(t, \zeta) \in [0, 1] \times \overline{\mathbb{D}}$, be the solution of the differential equation

$$(2.12) \quad \frac{d}{dt} X_\zeta(t) = V(t, X_\zeta(t)), \quad X_\zeta(1) = \zeta.$$

Put

$$(2.13) \quad \varphi_t(\zeta) = X_\zeta(t) \quad \text{and} \quad \varphi(t, \zeta) = (t, X_\zeta(t)), \quad (t, \zeta) \in [0, 1] \times \overline{\mathbb{D}}.$$

Then φ is a diffeomorphism of $[0, 1] \times \overline{\mathbb{D}}$ onto itself, φ_1 is the identity and the graphs of the f_j are integral curves, hence $\varphi_t(f_j(1)) = f_j(t)$. Moreover, for each t , φ_t is the identity near $\partial \mathbb{D}$ since $V(t, \zeta) = (1, 0)$ for (t, ζ) in a neighbourhood of $[0, 1] \times \partial \mathbb{D}$. \square

We need the following slightly stronger version of Proposition 2.3.

PROPOSITION 2.4. *Let $f^s : [0, 1] \rightarrow \mathfrak{P}_n$, $s \in [0, 1]$, be a free isotopy of geometric braids with variable base point $E_{n,s}$. Then there exists a continuous family*

$$\varphi_t^s \in \text{Hom}^+(\overline{\mathbb{D}}; \partial \mathbb{D}), \quad (t, s) \in [0, 1] \times [0, 1],$$

such that for $s \in [0, 1]$ the family φ_t^s , $t \in [0, 1]$, is a parametrizing isotopy for f^s , i.e. for the base point $E_{n,s}$ of f^s we have

$$\varphi_t^s(E_{n,s}) = f^s(t), \quad t \in [0, 1].$$

If $\varphi_t, t \in [0, 1]$, is a given parametrizing isotopy for f^0 then the family φ_t^s can be chosen so that $\varphi_t^0 = \varphi_t$.

If the mappings

$$[0, 1] \times [0, 1] \ni (t, s) \rightarrow f^s(t)$$

and

$$[0, 1] \times \overline{\mathbb{D}} \ni (s, \zeta) \rightarrow \varphi_t(\zeta),$$

are smooth, then the mapping

$$[0, 1] \times [0, 1] \times \overline{\mathbb{D}} \ni (t, s, \zeta) \rightarrow \varphi_t^s(\zeta),$$

can be chosen to be smooth.

Sketch of Proof. The proposition can be proved in the same way as Proposition 2.3. Construct a vector field $V(t, s, \zeta)$ in $[0, 1] \times [0, 1] \times \overline{\mathbb{D}}$ with the following properties.

- The projection of each vector $V(t, s, \zeta)$ to the first two real components is the unit vector in the t -direction;
- For each s the set $\{(t, s, f_s(t)) : t \in [0, 1]\}$ is the union of n integral curves of the vector field;
- For $s = 0$ the vector field $V(t, 0, \zeta)$ equals $\frac{d}{dt}\varphi_t$.

Assume first that $\varphi_t(\zeta) = \zeta$ for all t . We construct a vector field by repeating the construction in the proof of Proposition 2.3. It remains to put $\varphi_t^s(\zeta)$, $t \in [0, 1]$, equal to the time t mapping of the flow of the vector field (directed backwards from 1 to t , i.e. the initial value condition is $\varphi_1^s(\zeta) = \zeta$ for all s and ζ).

In the general situation we apply the previous arguments to the situation when φ_t is replaced by the identity, and $f^s(t)$ is replaced by $\varphi_t^{-1}(f_s(t))$. By the arguments given above we obtain a family $\tilde{\varphi}_t^s$. The composition $\varphi_t^s = \varphi_t \circ \tilde{\varphi}_t^s$ satisfies the requirements of the proposition in the general situation. \square

Remark 2.1. Put $E_n = E_{n,0}$. We denote by $\psi^s \in \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D})$, $s \in [0, 1]$, a continuous family such that $\psi^s(E_n) = E_{n,s}$, $s \in [0, 1]$. (It can be interpreted as parametrizing isotopy for the family of base points $E_{n,s}$.) Then the family

$$g_s(t) = ((\psi^s)^{-1} \circ \varphi_t^s \circ \psi^s)(E_n), \quad t \in [0, 1],$$

is an isotopy of braids with fixed base point E_n so that $g_0(t) = f_0(t)$, $t \in [0, 1]$, and $f_1(t) = \psi^s \circ g_1(t) \circ (\psi^s)^{-1}$, $t \in [0, 1]$, in other words f_1 is conjugate to g_1 .

Beltrami differentials and quadratic differentials. (For more details see e.g. [2], [26], [29], [36].) Let X be a Riemann surface. A Beltrami differential μ on X assigns to each chart on X with holomorphic coordinates z an essentially bounded measurable function $\mu(z)$ so that $\mu(z) \frac{\bar{dz}}{dz}$ is invariant under holomorphic coordinate changes. In other words, the functions $\mu_1(z)$ and $\mu_2(\zeta)$ associated to local coordinates ζ and $z(\zeta)$ are related by the equation

$$(2.14) \quad \mu_1(z(\zeta)) \left(\frac{dz}{d\zeta} \right)^{-1} \overline{\left(\frac{dz}{d\zeta} \right)} = \mu_2(\zeta).$$

(Hence, Beltrami differentials can be interpreted as sections in the bundle $\kappa^{-1} \otimes \bar{\kappa}$ for the cotangent bundle κ of X .)

By an abuse of notation we will denote the Beltrami differential on X by μ and write $\mu = \mu(z) \frac{d\bar{z}}{dz}$, where the left hand side denotes the globally defined Beltrami differential and $\mu(z)$ on the right hand side is a representing function in local coordinates z . The value $|\mu(z)|$ is invariant under holomorphic coordinate change. Put $\|\mu\|_\infty \stackrel{\text{def}}{=} \sup_X |\mu|$.

Let X and Y be Riemann surfaces and let $\varphi : X \rightarrow Y$ be a smooth orientation preserving homeomorphism. Let z be coordinates near a point of X and let ζ be coordinates on Y near its image under φ . We write $\zeta = \varphi(z)$ in these coordinates. Consider the function $\frac{\bar{\partial}\varphi}{\partial\varphi} = \frac{\frac{\partial}{\partial\bar{z}}\varphi}{\frac{\partial}{\partial z}\varphi}$ in these coordinates. Since for the Jacobian $\mathcal{J}(z)$ of $\varphi(z)$, $\mathcal{J}^2(z) = \left| \frac{\partial}{\partial z}\varphi(z) \right|^2 - \left| \frac{\partial}{\partial\bar{z}}\varphi(z) \right|^2 > 0$, the denominator $\partial\varphi$ does not vanish and $\left| \frac{\bar{\partial}\varphi}{\partial\varphi} \right| < 1$. The expression $\frac{\frac{\partial}{\partial\bar{z}}\varphi(z) d\bar{z}}{\frac{\partial}{\partial z}\varphi(z) dz}$ is invariant under holomorphic coordinate change on X and on Y . It defines a Beltrami differential μ_φ on X . The mapping φ is called quasiconformal if $\|\mu_\varphi\|_\infty < 1$. If X and Y are compact this is automatically so. The condition that φ is differentiable can be weakened. For a detailed account on quasiconformal mappings we refer to [2].

The quasiconformal dilatation of the mapping φ is defined as $K(\varphi) = \frac{1+\|\mu_\varphi\|_\infty}{1-\|\mu_\varphi\|_\infty}$ if φ is quasiconformal and as ∞ otherwise.

A meromorphic (respectively, holomorphic) quadratic differential ϕ on X assigns to each chart on X with holomorphic coordinates z a meromorphic (respectively, holomorphic) function which by abusing notation we denote by $\phi(z)$, such that $\phi(z)(dz)^2$ is invariant under holomorphic changes of coordinates. In other words, the functions $\phi_1(z)$ and $\phi_2(\zeta)$ associated to local coordinates ζ and $z(\zeta)$ are related by the formula

$$(2.15) \quad \phi_1(z(\zeta)) \left(\frac{dz}{d\zeta} \right)^2 = \phi_2(\zeta).$$

Holomorphic quadratic differentials on X can be regarded as holomorphic sections of the bundle κ^2 . By an abuse of notation we will write $\phi = \phi(z) dz^2$, where the left hand side denotes the quadratic differential and $\phi(z)$ on the right hand side denotes the meromorphic function that represents ϕ in local coordinates z . (The value of ϕ on a vector $v \in T_z X$ is $\phi(z)(dz(v))^2$.)

For a holomorphic quadratic differential on X and a small open subset $\mathcal{U} \subset X$ the integral $\iint_{\mathcal{U}} |\phi| = \frac{1}{2} \iint_{\mathcal{U}} |\phi(z)| dz \wedge d\bar{z}$ is invariant under holomorphic coordinate changes. Hence, $\|\phi\|_1 \stackrel{\text{def}}{=} \iint_X |\phi|$ is well defined. ϕ is called integrable on X if this integral is finite.

Consider meromorphic quadratic differentials on a Riemann surface X . The poles and zeros of the quadratic differential (i.e. of the meromorphic functions in local coordinates associated to it) are called its singularities. The singularities of integrable quadratic differentials are zeros or simple poles. The set of integrable meromorphic quadratic differentials on X with norm $\|\cdot\|_1$ forms a complex Banach space.

The following fact is an immediate corollary of the definitions. For each meromorphic quadratic differential ϕ and any number $k \in (0, 1)$ the object $k \frac{|\phi|}{\phi} = k \frac{\bar{\phi}}{|\phi|}$

(given in local coordinates z by $k \frac{\bar{\phi}(z)}{|\phi(z)|}$ with a meromorphic function $\phi(z)$) is a Beltrami differential of norm $K = \frac{1+k}{1-k}$.

A meromorphic quadratic differential ϕ produces singular foliations as follows. For each non-singular $z \in X$ the condition $\phi(z)(dz)^2 > 0$ defines a (non-oriented) line element: if for a representing function $\phi(z)$ in local coordinates z we have $\frac{\phi(z)}{|\phi(z)|} = e^{i\theta}$, we obtain the line $t \rightarrow t e^{-i\frac{\theta}{2}}$, $t \in \mathbb{R}$. The maximal integral curves of this line field are called the horizontal trajectories of ϕ . They define a foliation with singularities at the zeros and poles of the quadratic differential. For each $\theta \in [0, 2\pi)$ the θ -trajectories are the horizontal trajectories of the quadratic differential $e^{-i\theta} \phi$. The π -trajectories are also called vertical trajectories (the respective line field is determined by the condition $\phi(z)(dz)^2 < 0$). The horizontal and the vertical foliation have the same set of singular points and are orthogonal at regular points. In a neighbourhood of a regular point there are local holomorphic coordinates z of the quadratic differential $\phi(\zeta) d\zeta^2$, called distinguished coordinates, with the following properties. The coordinates vanish at the given regular point and in these coordinates the quadratic differential has the form dz^2 . (These coordinates are obtained as an integral of a branch of $\sqrt{\phi(\zeta)}$. Hence, the distinguished coordinates in a neighbourhood of a point are uniquely defined up to sign.) In distinguished coordinates near a regular point the horizontal (respectively, vertical) trajectories are subsets of the horizontal (respectively, vertical) lines. Note that vice versa, any pair of singular foliations on a closed Riemann surface with equal set of singularities, which are orthogonal at regular points, defines a quadratic differential.

Near a singular point there are holomorphic coordinates z vanishing at the point in which the quadratic differential has the form

$$(2.16) \quad \phi(z) dz^2 = \left(\frac{a+2}{2} \right)^2 z^a dz^2$$

for some integer a . The coordinates (2.16) are uniquely defined up to multiplication by an $(a+2)$ -nd root of unity and are called distinguished coordinates. The number a is called the order of the point. Distinguished coordinates near regular points have the form (2.16) with $a = 0$. We will consider quadratic differentials with at worst poles of order one, i.e. $a \geq -1$.

Teichmüller theorem (closed surfaces). (For more details, see [3], [6].)

THEOREM 2.2. *Let X and Y be closed Riemann surfaces of genus $g \geq 2$, and let $\varphi : X \rightarrow Y$ be a homeomorphism. Then there is a unique homeomorphism isotopic to φ with smallest quasiconformal dilatation. This homeomorphism is either conformal or its Beltrami differential has the form $k \cdot \frac{|\phi|}{\phi}$ for a holomorphic quadratic differential ϕ on X and a constant $k \in (0, 1)$. ϕ is unique up to multiplication by a positive constant.*

A homeomorphism with the latter property is called a Teichmüller mapping and ϕ is called its quadratic differential.

Let X and Y be closed Riemann surfaces or Riemann surfaces of first kind. Suppose $\varphi : X \rightarrow Y$ is a Teichmüller mapping with quadratic differential ϕ and constant k . Then the inverse mapping φ^{-1} is again a Teichmüller mapping with quadratic differential denoted by $-\psi$ and constant k . The order of ϕ at a point z is the same as the order of $-\psi$ at the image $\varphi(z)$.

There are distinguished coordinates z for ϕ near a point $z_0 \in X$ which vanish at z_0 and distinguished coordinates ζ for $-\psi$ near the image point $\varphi(z_0) \in Y$ which vanish at $\varphi(z_0)$ so that the mapping φ has the form

$$(2.17) \quad \zeta = \left(\frac{z^{a+2} + 2k|z|^{a+2} + k^2 \bar{z}^{a+2}}{1 - k^2} \right)^{\frac{1}{a+2}}$$

with $\zeta > 0$ for $z > 0$. If $a = 0$ this is equivalent to $\zeta = \xi + i\eta = K^{\frac{1}{2}}x + iK^{-\frac{1}{2}}y$ with $z = x + iy$ and $K = \frac{1+k}{1-k}$ being the quasiconformal dilatation.

We will describe now an idea of Ahlfors which allows to reduce questions concerning punctured Riemann surfaces to the related questions concerning closed Riemann surfaces. Ahlfors used it for a proof of Teichmüller's theorem for punctured Riemann surfaces. It can also be used for studying the entropy of certain homeomorphisms of punctured Riemann surfaces.

Let X and Y be closed Riemann surfaces, both with a set of m distinguished points. (Using the identification of $\text{Hom}^+(X, \emptyset, E)$ with $\text{Hom}^+(X \setminus E, \emptyset, \emptyset)$ we may think about two m -punctured Riemann surfaces.) Assume $2g - 2 + m > 0$, so that the universal covering of the m -punctured surfaces equals \mathbb{C}_+ . Associate to X and Y closed Riemann surfaces \tilde{X} and \tilde{Y} which are holomorphic simple branched coverings of X and Y with branch locus at the set of distinguished points and have genus at least two. If the number of punctures is even (and not zero) and $2g + \frac{m}{2} - 1 \geq 2$ (i.e. either $g \geq 1$ or $m \geq 6$) then one can take a double branched covering with branch points at the distinguished points. Otherwise the construction has to be modified. Take a homeomorphism $\varphi : X \rightarrow Y$ which maps the set of distinguished points of X to the set of distinguished points of Y . A lift $\tilde{\varphi}$ of φ , $\tilde{\varphi} : \tilde{X} \rightarrow \tilde{Y}$, is a homeomorphism between closed surfaces which has some "additional symmetries". Vice versa, a homeomorphism between \tilde{X} and \tilde{Y} with such symmetries is a lift of a homeomorphism between X and Y which maps the set of distinguished points in X to the set of distinguished points in Y .

Teichmüller's theorem also applies in the situation of homeomorphisms with "additional symmetry" between closed Riemann surfaces. One obtains quadratic differentials on \tilde{X} with a "symmetry", and one obtains quadratic differentials on X which lift to the mentioned quadratic differentials on \tilde{X} . This implies that the quadratic differentials on X have at most simple poles at the branch locus (a simple calculation using the behaviour of quadratic differentials under coordinate changes). Hence Teichmüller's theorem remains true for homeomorphisms between closed surfaces with distinguished points, if instead of holomorphic quadratic differentials one considers meromorphic quadratic differentials on closed surfaces with at most simple poles at distinguished points. These are exactly the integrable meromorphic quadratic differentials.

Ahlfors also treats Riemann surfaces with finite-dimensional fundamental group of the second kind in a similar way (i.e. Riemann surfaces with finitely many boundary components, some of which may be points, some are continua). He uses extension of the homeomorphism to a homeomorphism between the doubles of the Riemann surfaces. We do not need this case here. For details see [3].

Teichmüller spaces. Let X be a connected Riemann surface of genus g , with ℓ boundary continua and m punctures. We always assume that the universal covering

of X equals \mathbb{C}_+ . (This excludes the Riemann sphere, \mathbb{C} , $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and tori (compact Riemann surfaces of genus 1).)

Let $w_j : X \rightarrow Y_j$, $j = 1, 2$, be two quasiconformal homeomorphisms onto Riemann surfaces (quasiconformal complex structures on X). They are called (Teichmüller) equivalent if there exists a conformal mapping $c : Y_1 \rightarrow Y_2$ such that $w_2^{-1} \circ c \circ w_1$ is isotopic to the identity by an isotopy which fixes the set of boundary continua pointwise (if any). We denote the equivalence class of a quasiconformal complex structure $w : X \rightarrow Y$ on X by $[w]$. The set of equivalence classes is the Teichmüller space $\mathcal{T}(X)$. Equip the Teichmüller space with the Teichmüller metric $d_{\mathcal{T}}$,

$$(2.18) \quad d_{\mathcal{T}}([w_1], [w_2]) \stackrel{\text{def}}{=} \inf \left\{ \frac{1}{2} \log K(v_2 \circ v_1^{-1}) : v_1 \in [w_1], v_2 \in [w_2] \right\}.$$

Note that the infimum is equal to the following

$$(2.19) \quad \inf \left\{ \frac{1}{2} \log K(g) : w_2^{-1} \circ g \circ w_1 : X \rightarrow X \text{ is isotopic to the identity} \right. \\ \left. \text{fixing the boundary continua pointwise} \right\}.$$

For a quasi-conformal homeomorphism w with $[w] \in \mathcal{T}(X)$ the space $\mathcal{T}(w(X))$ is canonically isometric to $\mathcal{T}(X)$. We choose a reference Riemann surface X with m punctures and ℓ boundary continua and write $\mathcal{T}(g, m, \ell)$.

We also need the Fuchsian model of the Teichmüller space. Recall that a Fuchsian group is a discrete subgroup of $\text{PSL}(2, \mathbb{R})$. $\text{PSL}(2, \mathbb{R})$ is identified with the group of Möbius transformations which map the upper half-plane \mathbb{C}_+ onto itself.

Consider Riemann surfaces X and Y whose universal coverings equals \mathbb{C}_+ . Represent X and Y as quotients of the upper half-plane \mathbb{C}_+ by the action of Fuchsian groups Γ and Γ_1 , i.e. $X \cong \mathbb{C}_+ / \Gamma$, $Y \cong \mathbb{C}_+ / \Gamma_1$. Then homeomorphisms w from X to Y lift to self-homeomorphism \tilde{w} of the upper half-plane such that

$$(2.20) \quad \text{for each } \gamma \in \Gamma \text{ there is } \gamma_1^{\tilde{w}} \in \Gamma_1 \text{ such that } \gamma \circ \tilde{w} = \tilde{w} \circ \gamma_1^{\tilde{w}},$$

and self-homeomorphisms of \mathbb{C}_+ with this property project to homeomorphisms from X onto Y .

Beltrami differentials and quadratic differentials on X lift to Beltrami differentials $\tilde{\mu}$ and quadratic differentials $\tilde{\varphi}$ on \mathbb{C}_+ with the invariance property

$$(2.21) \quad \tilde{\mu} \circ \gamma = \tilde{\mu} \frac{\gamma'}{\gamma'} \quad \text{for } \gamma \in \Gamma \quad (\text{automorphic } (-1, 1)\text{-forms}),$$

$$(2.22) \quad \tilde{\varphi} \circ \gamma \cdot \gamma'^2 = \tilde{\varphi} \quad \text{for } \gamma \in \Gamma \quad (\text{automorphic 2-forms}).$$

If a self-homeomorphism of \mathbb{C}_+ satisfies (2.20) then its Beltrami differential satisfies (2.21). If the homeomorphism is extremal its quadratic differential satisfies (2.22).

A self-homeomorphism of the closed upper half-plane $\bar{\mathbb{C}}_+$ (a self-homeomorphism of the Riemann sphere \mathbb{P}^1 , respectively) is called normalized if it maps 0 to 0, 1 to 1 and ∞ to ∞ . Let Γ be a Fuchsian group. Denote by $Q_{\text{norm}}(\Gamma)$ the set of normalized quasiconformal self-homeomorphisms of \mathbb{C}_+ that satisfy (2.20) for another Fuchsian group Γ_1 . Note that each quasiconformal self-homeomorphism of \mathbb{C}_+ extends to a self-homeomorphism of $\bar{\mathbb{C}}_+$. The Beltrami differentials on \mathbb{C}_+ that satisfy (2.21) are in one-to-one correspondence to elements of $Q_{\text{norm}}(\Gamma)$. Indeed,

associate to each Beltrami differential μ on \mathbb{C}_+ the Beltrami differential $\hat{\mu}$ on \mathbb{C} , for which $\hat{\mu}(z) = \mu(z)$, $z \in \mathbb{C}_+$, $\hat{\mu}(z) = \bar{\mu}(\bar{z})$, $z \in \mathbb{C}_-$ (\mathbb{C}_- denotes the lower half-plane). There is a unique normalized solution w of the equation $w_{\bar{z}} = \hat{\mu}(z) w_z$ on the complex plane. It maps \mathbb{C}_+ onto itself. Its restriction to \mathbb{C}_+ is denoted by w^μ . Let $\Gamma_1 = \Gamma^\mu$ be the group $w^\mu \circ \gamma \circ (w^\mu)^{-1}$, $\gamma \in \Gamma$. This is a Fuchsian group. If μ satisfies (2.21) then w^μ satisfies (2.20).

Two elements of $Q_{\text{norm}}(\Gamma)$ are called equivalent iff their restrictions to the real axis coincide. The Teichmüller space $\mathcal{T}(\Gamma)$ is defined as set of equivalence classes of elements of $Q_{\text{norm}}(\Gamma)$. Note that if $\Gamma^\mu = \Gamma^\nu$ then $w^\mu, w^\nu \in Q_{\text{norm}}(\Gamma)$ are equivalent iff the mappings w^μ, w^ν of \mathbb{C}_+ induce Teichmüller equivalent mappings W^μ, W^ν on $X = \mathbb{C}_+/\Gamma$.

Let μ be a Beltrami differential X , let $\tilde{\mu}$ be its lift to \mathbb{C}_+ and let $w^{\tilde{\mu}}$ be the normalized solution of the Beltrami equation on \mathbb{C}_+ for $\tilde{\mu}$. The projection of $w^{\tilde{\mu}}$ to X is denoted by W^μ . For later use we give the following definition.

DEFINITION 2.1. *For each Beltrami differential μ on X the homeomorphism $W^\mu \in QC(X)$ is called the normalized solution of the Beltrami equation on X for the Beltrami differential μ .*

Also, we assign to μ the element $[W^\mu]$ of the Teichmüller space $\mathcal{T}(X)$. We use the notation $\{\mu\}$ for $[W^\mu]$. The obtained mapping from $\mathcal{T}(\Gamma)$ into $\mathcal{T}(X)$ is a bijection.

Let now X be a Riemann surface of genus g with m punctures and no boundary continuum. Let X^{Cl} be its closure (obtained by filling the punctures). We assume that the universal covering of X equals \mathbb{C}_+ . The Teichmüller space is denoted by $\mathcal{T}(X) \cong \mathcal{T}(g, m, 0)$. Instead of $\mathcal{T}(g, m, 0)$ we will write $\mathcal{T}(g, m)$. Teichmüller's theorem implies the following.

Denote by $QC(X)$ the set of quasiconformal homeomorphisms of X onto another Riemann surface. The mapping $QC(X) \xrightarrow{[\cdot]} \mathcal{T}(X)$ assigns to each element $w \in QC(X)$ its class $[w]$ in the Teichmüller space $\mathcal{T}(X)$. The Teichmüller space is equipped with the Teichmüller metric $d_{\mathcal{T}}$. Associate to each non-trivial class $[w] \in \mathcal{T}(X)$ the unique extremal quasiconformal homeomorphism in this class. The extremal quasiconformal homeomorphisms are in bijection to Beltrami differentials of the form $k \frac{|\phi|}{\phi}$ on X where k is a constant in $(0, 1)$ and ϕ is a holomorphic quadratic differential on X which extends to a meromorphic quadratic differential on X^{Cl} with at most simple poles at the punctures. Hence, extremal quasiconformal homeomorphisms are in bijection with integrable meromorphic quadratic differentials $k \frac{\phi}{\|\phi\|_1}$ on X^{Cl} of norm k less than 1 whose restriction to X is holomorphic. The real dimension of the space of integrable holomorphic quadratic differentials on X is equal to $6g - 6 + 2m$. It can be proved that the Teichmüller space $\mathcal{T}(X)$ is homeomorphic to the unit ball in the Banach space of such quadratic differentials.

There is a unique conformal structure on $\mathcal{T}(g, m)$ with the following property. Take any family of complex structures in $QC(X)$ whose Beltrami differentials depend holomorphically on certain complex parameters. Then the equivalence classes in $\mathcal{T}(X)$ depend holomorphically on the complex parameters.

The explicit construction uses the Fuchsian model (see [7]). Write $X = \mathbb{C}_+/\Gamma$. Let μ be a Beltrami differential on \mathbb{C}_+ satisfying (2.21). Let w_μ be the unique normalized solution of the Beltrami equation on the Riemann sphere with Beltrami

coefficient equal to μ on \mathbb{C}_+ and equal to 0 on \mathbb{C}_- . The mapping w_μ does not map \mathbb{C}_+ onto \mathbb{C}_+ , but if the Beltrami differentials depend holomorphically on complex parameters then the mappings w_μ depend holomorphically on them. The mappings w_μ are conformal on the lower half-plane \mathbb{C}_- . Moreover, for two Beltrami differentials ν and μ the equality $w^\mu = w^\nu$ holds on \mathbb{R} iff the equality $w_\mu = w_\nu$ holds on \mathbb{R} (and hence the latter equality holds on \mathbb{C}_-). Consider the Schwarzian derivative $\mathfrak{S}(w_\mu | \mathbb{C}_-)$. (For a locally conformal mapping f on an open set in \mathbb{P}^1 the Schwarzian derivative is defined as $\mathfrak{S}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2$.)

The Schwarzian derivatives of Möbius transformations equal zero and $\mathfrak{S}(f \circ g) = \mathfrak{S}(f) \circ g \cdot (g')^2 + \mathfrak{S}(g)$ for locally conformal mappings f and g . This implies that $\phi_\mu = \mathfrak{S}(w_\mu | \mathbb{C}_-)$ is a quadratic differential on \mathbb{C}_- which satisfies (2.22) (with respect to Γ acting on \mathbb{C}_-). If μ depends holomorphically on parameters then so does ϕ_μ . We obtained a map $\mu \rightarrow \phi_\mu = \mathfrak{S}(w_\mu | \mathbb{C}_-)$ from the set of Beltrami differentials on \mathbb{C}_+ satisfying (2.21) to the space of holomorphic quadratic differentials on \mathbb{C}_- satisfying the condition

$$(2.23) \quad \phi_\mu = \phi_\nu \quad \text{if} \quad w^\mu, w^\nu \in Q_{\text{norm}}(\Gamma) \text{ are equivalent.}$$

Note that for each holomorphic function f on a simply connected domain in the complex plane there is a meromorphic function w in the domain, unique up to a Möbius transformation for which $\mathfrak{S}(w) = f$. There is an explicit way to find such a function.

Consider the Banach space $B(\mathbb{C}_-, \Gamma)$ of holomorphic quadratic differentials in \mathbb{C}_- satisfying (2.23) with norm $\|\varphi\| = \sup |y^2 \varphi(z)|$. The mapping

$$(2.24) \quad \mathcal{T}(X) \cong \mathcal{T}(\Gamma) \ni \{\mu\} \rightarrow \phi_\mu$$

defines a homeomorphism of $\mathcal{T}(X)$ onto an open subset of the unit ball of $B(\mathbb{C}_-, \Gamma)$. This homeomorphism is called Bers embedding. The complex structure on $\mathcal{T}(X)$ induced by this homeomorphism from $B(\mathbb{C}_-, \Gamma)$ is the desired one. Notice, that for Beltrami differentials μ on X depending holomorphically on parameters, the homeomorphisms W^μ may not have this property, but their Teichmüller classes $[W^\mu] = \{\mu\}$ have this property.

Teichmüller discs. Let X be a Riemann surface of genus g with m punctures with universal covering \mathbb{C}_+ . Write $X = \mathbb{C}_+/\Gamma$ for a Fuchsian group Γ . Let ϕ be a meromorphic quadratic differential on X which is holomorphic on X^{cl} except, maybe, at some punctures, where it may have simple poles. For each $z = r e^{i\theta} \in \mathbb{D}$ we consider the Beltrami differential $\mu_z = z \frac{|\phi|}{\phi} = r \frac{|e^{-i\theta} \phi|}{e^{-i\theta} \phi}$. Each Beltrami differential μ_z defines a unique element $w^{\mu_z} \in Q_{\text{norm}}(\Gamma)$, equivalently a unique normalized solution $W^{\mu_z} \in QC(X)$ associated to μ_z , and a unique Teichmüller class $[W^{\mu_z}] = \{\mu_z\} \in \mathcal{T}(X) \cong \mathcal{T}(\Gamma)$. The mapping

$$(2.25) \quad \mathbb{D} \ni z \rightarrow \{\mu_z\} \in \mathcal{T}(X)$$

is a holomorphic embedding. For each $z \neq 0$ the homeomorphism W^{μ_z} is a Teichmüller map. Let $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$. One can show that $W^{\mu_{z_2}} \circ (W^{\mu_{z_1}})^{-1}$ is a Teichmüller mapping. The absolute value of its Beltrami differential equals

$$(2.26) \quad K(W^{\mu_{z_2}} \circ (W^{\mu_{z_1}})^{-1}) = \left| \frac{z_2 - z_1}{1 - \bar{z}_2 z_1} \right|.$$

Hence, $\frac{1}{2} \log K(W^{\mu_{z_2}} \circ (W^{\mu_{z_1}})^{-1})$ is equal to the distance of z_1 and z_2 in the Poincaré metric. (For details see, e.g. [26].) Thus the mapping $\mathbb{D} \ni z \rightarrow \{\mu_z\} \in \mathcal{T}(X)$ is an isometric holomorphic embedding of the disc with Poincaré metric into Teichmüller space with Teichmüller metric. It is called the Teichmüller disc associated to ϕ . We will denote its image in $\mathcal{T}(X)$ by \mathcal{D}_ϕ . Note that the embedding is proper.

The modular group. A quasiconformal self-homeomorphism φ of a closed Riemann surface of genus g with a set E of m distinguished points induces a mapping φ^* of the Teichmüller space $\mathcal{T}(X) \cong \mathcal{T}(g, m)$ to itself. It is defined as follows. For each homeomorphism $w \in QC(X)$ the composition $w \circ \varphi : X \rightarrow X$ is another quasiconformal homeomorphism with the set E of distinguished points. Its class $[w \circ \varphi] \in \mathcal{T}(g, m)$ depends only on the class $[w]$ of w . Put $\varphi^*([w]) = [w \circ \varphi]$. The mapping φ^* is an isometry on the Teichmüller space $\mathcal{T}(g, m)$. Moreover, it maps $\mathcal{T}(g, m)$ biholomorphically onto itself. The mapping φ^* is called the modular transformation of φ . The set of modular transformations forms a group, called the modular group. It is often denoted by $\text{Mod}(g, m)$. Isotopic homeomorphisms with the set E of distinguished points have the same modular transformation. The modular group is isomorphic to the mapping class group $\mathfrak{M}(X \setminus E) = \mathfrak{M}(X, \emptyset, E)$. The quotient $\mathcal{T}(g, m) / \text{Mod}(g, m)$ can be identified with the Riemann space of conformally equivalent complex structures, i.e. with the moduli space of Riemann surfaces of genus g with m punctures.

Royden's theorem. The following deep theorem of Royden [32] has many applications.

THEOREM 2.3 (Royden). *The Teichmüller metric on the Teichmüller space $\mathcal{T}(g, m)$ is equal to the Kobayashi metric. In other words, a holomorphic mapping from the unit disc \mathbb{D} with Poincaré metric $\frac{|dz|}{1-|z|^2}$ into $\mathcal{T}(g, m)$ with Teichmüller metric is a contraction. (Equivalently, a holomorphic map from \mathbb{C}_+ with hyperbolic metric $\frac{|dz|}{2y}$ into $\mathcal{T}(g, m)$ is a contraction.)*

Configuration space and Teichmüller space. The following explanation will be needed to relate geometric n -braids to pathes in the Teichmüller space $\mathcal{T}(0, n+1)$ of the Riemann sphere with $n+1$ punctures. The reference Riemann surface will be $X = \mathbb{C} \setminus E_n^0 = \mathbb{P}^1 \setminus (\{\infty\} \cup E_n^0)$, where $E_n^0 = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$. Let $E_n^1 \subset \mathbb{C}$ be another set containing exactly n points. For a homeomorphism from $\mathbb{C} \setminus E_n^0$ onto $\mathbb{C} \setminus E_n^1$ we will use the same notation for this homeomorphism and for the extension of this homeomorphism to a self-homeomorphism of \mathbb{P}^1 which maps E_n^0 to E_n^1 and ∞ to ∞ . Denote by $\mathcal{C}(0, n+1)$ the set of orientation preserving self-diffeomorphisms w of \mathbb{P}^1 which fix E_n^0 setwise and are equal to the identity outside the unit disc \mathbb{D} . For $w \in \mathcal{C}(0, n+1)$ the mapping $w|_{\mathbb{C} \setminus E_n^0}$ is considered as conformal structure on $\mathbb{C} \setminus E_n^0$. Equip $\mathcal{C}(0, n+1)$ with the topology of C^1 convergence (actually with C^1 topology of restrictions $w|_{\overline{\mathbb{D}}}$, $w \in \mathcal{C}(0, n+1)$). Each element $w \in \mathcal{C}(0, n+1)$ is a quasiconformal mapping. Indeed, the quasiconformal dilatation is $\sup_{\overline{\mathbb{D}}} \frac{|\frac{\partial}{\partial \bar{z}} w| + |\frac{\partial}{\partial z} w|}{|\frac{\partial}{\partial \bar{z}} w| - |\frac{\partial}{\partial z} w|}$.

This is finite, since the square of the Jacobian $\mathcal{J}^2 = |\frac{\partial}{\partial z} w|^2 - |\frac{\partial}{\partial \bar{z}} w|^2$ is uniformly bounded from below on the compact set $\overline{\mathbb{D}}$. The mapping $\mathcal{C}(0, n+1) \xrightarrow{[\cdot]} \mathcal{T}(0, n+1)$, which assigns to each element $w \in \mathcal{C}(0, n+1)$ its class $[w]$, is continuous. Indeed,

for two elements $w_1, w_2 \in \mathcal{C}(0, n+1)$ the Teichmüller distance $d_{\mathcal{T}}([w_1], [w_2])$ does not exceed

$$(2.27) \quad \frac{1}{2} \sup_{\mathbb{D}} \log \frac{1 + \|\mu_{w_2 \circ w_1^{-1}}\|}{1 - \|\mu_{w_2 \circ w_1^{-1}}\|}.$$

The formula

$$(2.28) \quad \mu_{w_2 \circ w_1^{-1}} \circ w_1 = \frac{\frac{\partial}{\partial z} w_1}{\frac{\partial}{\partial z} w_1} \cdot \frac{\mu_{w_2} - \mu_{w_1}}{1 - \mu_{w_2} \mu_{w_1}}$$

implies for w_2 close to w_1 in C^1 norm an estimate from above of $d_{\mathcal{T}}([w_1], [w_2])$ by $\text{const} \|w_2 - w_1\|_{C^1(\mathbb{D})}$ with a constant depending on w_1 .

Each self-homeomorphism ψ of \mathbb{C} acts diagonally on the configuration space $C_n(\mathbb{C})$. Denote this action again by ψ :

$$(2.29) \quad \psi : (z_1, \dots, z_n) \rightarrow (\psi(z_1), \dots, \psi(z_n)), \quad (z_1, \dots, z_n) \in C_n(\mathbb{C}).$$

The action descends to the symmetrized configuration space $C_n(\mathbb{C})/\mathcal{S}_n$.

Let \mathcal{A} be the set of complex affine mappings on the complex plane. Each element $\mathbf{a} \in \mathcal{A}$ has the form $\mathbf{a}(z) = az + b$, $z \in \mathbb{C}$. Here $b \in \mathbb{C}$ and $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ are constants. \mathcal{A} has the complex structure of $\mathbb{C}^* \times \mathbb{C}$. It forms a group under composition.

Denote by $C_n(\mathbb{C})/\mathcal{A}$ the quotient of the configuration space $C_n(\mathbb{C})$ by the diagonal action of the group. Each element of $C_n(\mathbb{C})/\mathcal{A}$ contains a unique representative of the form $(0, \frac{1}{n}, z'_3, \dots, z'_n)$, where the z'_j are mutually distinct and not equal to 0 or $\frac{1}{n}$. For $z = (z_1, z_2, \dots, z_n) \in C_n(\mathbb{C})$ there is a unique element $\mathbf{a} = \mathbf{a}_z \in \mathcal{A}$ for which $\mathbf{a}(z) = (\mathbf{a}(z_1), \dots, \mathbf{a}(z_n))$ has this form. This element is

$$(2.30) \quad \mathbf{a}_z(\zeta) = \frac{1}{n} \frac{\zeta - z_1}{z_2 - z_1} = \frac{1}{n} \frac{1}{z_2 - z_1} \zeta + \frac{1}{n} \frac{-z_1}{z_2 - z_1}, \quad \zeta \in \mathbb{C}.$$

Hence $C_n(\mathbb{C})/\mathcal{A}$ can be identified with

$$C_{n-2}(\mathbb{C} \setminus \{0, \frac{1}{n}\}) = \{(z_3, \dots, z_n) \in (\mathbb{C} \setminus \{0, \frac{1}{n}\})^{n-2} : z_i \neq z_j \text{ for } i \neq j\}$$

and $C_n(\mathbb{C})$ is isomorphic to $\mathcal{A} \times (C_n(\mathbb{C})/\mathcal{A}) \cong \mathbb{C}^* \times \mathbb{C} \times C_{n-2}(\mathbb{C} \setminus \{0, \frac{1}{n}\})$.

The canonical projection $\mathcal{P}_{\mathcal{A}} : C_n(\mathbb{C}) \rightarrow C_n(\mathbb{C})/\mathcal{A}$ is holomorphic.

Recall that for each self-homeomorphism of \mathbb{P}^1 , in particular, for each element w of $\mathcal{C}(0, n+1)$ the map $e_n(w)$ is defined by $e_n(w) = \left(w(0), w\left(\frac{1}{n}\right), \dots, w\left(\frac{n-1}{n}\right)\right)$. It defines a continuous mapping $e_n : \mathcal{C}(0, n+1) \rightarrow C_n(\mathbb{C})$. Recall also that two complex structures w_1, w_2 on $\mathbb{C} \setminus E_n^0$ are Teichmüller equivalent if there is a conformal mapping $c : w_1(\mathbb{C} \setminus E_n^0) \rightarrow w_2(\mathbb{C} \setminus E_n^0)$ such that $w_2^{-1} \circ c \circ w_1 : \mathbb{C} \setminus E_n^0 \rightarrow \mathbb{C} \setminus E_n^0$ is isotopic to the identity on $\mathbb{C} \setminus E_n^0$. Denote by the same letters c, w_1, w_2 the extensions of the previous mappings to \mathbb{P}^1 . The mapping c is a Möbius transformation that fixes ∞ , hence $c \in \mathcal{A}$. We have $e_n(w_2) = c(e_n(w_1))$. In other words, to each Teichmüller class $[w] \in \mathcal{T}(0, n+1)$ corresponds a unique element of $C_n(\mathbb{C})/\mathcal{A}$ denoted by $\mathcal{P}_{\mathcal{T}}([w])$. The mapping $\mathcal{P}_{\mathcal{T}} : \mathcal{T}(0, n+1) \rightarrow C_n(\mathbb{C})/\mathcal{A}$ is a holomorphic covering and $\mathcal{T}(0, n+1)$ is the universal covering of $C_n(\mathbb{C})/\mathcal{A}$ ([21]).

The following lemma is tautological.

LEMMA 2.1. *The following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{C}(0, n+1) & \xrightarrow{[\]} & \mathcal{T}(0, n+1) \\ \downarrow e_n & & \downarrow \mathcal{P}_\tau \\ C_n(\mathbb{C}) & \xrightarrow{\mathcal{P}_\mathcal{A}} & C_n(\mathbb{C})/\mathcal{A} \end{array}$$

All mappings in the diagram are continuous.

In the following simple but useful lemma we again identify homomorphisms between punctured surfaces and their extensions to \mathbb{P}^1 .

LEMMA 2.2. *Let E_n^1 and E_n^2 be subsets of \mathbb{C} , each containing exactly n points. Let $w_1 : \mathbb{C} \setminus E_n^0 \rightarrow \mathbb{C} \setminus E_n^1$ and $w_2 : \mathbb{C} \setminus E_n^0 \rightarrow \mathbb{C} \setminus E_n^2$ be Teichmüller equivalent homeomorphisms. If $w_1(0) = w_2(0)$ and $w_1(\frac{1}{n}) = w_2(\frac{1}{n})$, then $E_n^1 = E_n^2$ and w_1, w_2 are isotopic as mappings from $\mathbb{C} \setminus E_n^0$ onto $\mathbb{C} \setminus E_n^1$. In particular, $e_n \circ w_1 = e_n \circ w_2$.*

Proof. If for an affine map $c \in \mathcal{A}$ the mapping $w_2^{-1} \circ c \circ w_1$ is isotopic to the identity through self-homeomorphisms of $\mathbb{C} \setminus E_n^0$ then it fixes E_n^0 pointwise, i.e. $c \circ w_1 \mid E_n^0 = w_2 \mid E_n^0$. Since $w_1(0) = w_2(0)$, $w_1(\frac{1}{n}) = w_2(\frac{1}{n})$, c fixes two points in \mathbb{C} , hence c is the identity. Thus $E_n^1 = w_1(E_n^0) = w_2(E_n^0) = E_n^2$ and $w_2^{-1} \circ w_1$ is isotopic to the identity through self-homeomorphisms of $\mathbb{C} \setminus E_n^0$. In other words, there is a continuous family φ^t , $t \in [0, 1]$, of self-homeomorphisms of $\mathbb{C} \setminus E_n^0$ with $\varphi^0 = w_2^{-1} \circ w_1$ and $\varphi^1 = \text{id}$. The required isotopy is $w_2 \circ \varphi^t$, $t \in [0, 1]$. The equality $e_n \circ w_1 = e_n \circ w_2$ follows. \square

Thurston's classification of mapping classes. Thurston's interest in surface homeomorphisms was motivated by his geometrization conjecture. He considered a closed surface S and a self-homeomorphism φ of S . The mapping torus

$$([0, 1] \times S) / (0, x) \sim (1, \varphi(x))$$

is obtained by gluing the fiber over the point 0 of the cylinder $[0, 1] \times S$ to the fiber over 1 using the homeomorphism φ . Thurston observed that for a class of homeomorphisms, which he called pseudo-Anosov, the mapping torus admits a complete hyperbolic structure of finite volume. This was one of the eight geometric structures. Moreover, Thurston gave a classification of mapping classes of surface homeomorphisms. Here is Thurston's theorem on classification of mapping classes.

THEOREM 2.4 (Thurston [37]). *A self-homeomorphism of a closed surface which is not isotopic to a periodic one is either isotopic to a pseudo-Anosov homeomorphism or is reducible, but not both.*

A finite non-empty set of mutually disjoint Jordan curves $\{C_1, \dots, C_\alpha\}$ on a Riemann surface X will be called admissible if no C_i is homotopic to a point in X , or to a boundary continuum, or to a puncture, or to a C_j with $i \neq j$. Thurston calls a homeomorphism $\varphi : X \rightarrow X$ reduced by the set $\{C_1, \dots, C_\alpha\}$, if this set is admissible and

$$\varphi(C_1 \cup C_2 \cup \dots \cup C_\alpha) = C_1 \cup C_2 \cup \dots \cup C_\alpha.$$

A self-mapping φ of X is called reducible if it is isotopic to a reduced mapping and irreducible otherwise. A mapping class is called reducible if it consists of reducible maps. Similarly, a braid $b \in \mathcal{B}_n$ is called reducible, if its associated mapping class

$\mathfrak{m}_b = \Theta_n(b) \in \mathfrak{M}(\overline{\mathbb{D}}; \partial\mathbb{D}, E_n^0)$ is reducible and is called irreducible otherwise. A conjugacy class is called reducible if its representatives are reducible.

Bers [5] gave a proof of Thurston's Theorem from the point of view of Teichmüller theory and obtained a description of reducible mappings. The proof of our Theorem 1 makes explicit use of the technique developed by Bers. We will outline now the results of Bers' approach to Thurston's theory which we need.

Consider a surface S . We do not require that it is closed, and we allow S to be the union of more than one, but at most finitely many, connected components. A conformal structure on S is a homeomorphism w of S onto a Riemann surface. Let φ be a self-homeomorphism of S . Consider the extremal problem to find the following infimum

$$(2.31) \quad I(\varphi) \stackrel{\text{def}}{=} \inf \{K(w \circ \tilde{\varphi} \circ w^{-1}) : w : S \rightarrow w(S) \text{ is a conformal structure on } S, \tilde{\varphi} \text{ is free isotopic to } \varphi\}.$$

This extremal problem differs from Teichmüller's extremal problem by varying also the conformal structure. Notice that it is not required that the conformal structures are quasiconformal. If the infimum is realized on a pair (w_0, φ_0) , then $w_0 \circ \varphi_0 \circ w_0^{-1}$ is called absolutely extremal and w_0 is called a φ_0 -minimal conformal structure.

Denote by \mathfrak{m}_φ the mapping class of φ and by $\widehat{\mathfrak{m}}_\varphi$ its conjugacy class,

$$\widehat{\mathfrak{m}}_\varphi = \{\psi = w \circ \tilde{\varphi} \circ w^{-1} : w \text{ is a conformal structure on } S, \tilde{\varphi} \text{ is isotopic to } \varphi\}.$$

Then $I(\varphi)$ can be written as follows

$$I(\varphi) = \inf \{K(\psi) : \psi \in \widehat{\mathfrak{m}}_\varphi\}.$$

In the following we will consider connected surfaces S unless said otherwise. A self-homeomorphism φ of S is periodic if φ^n is the identity on S for a natural number n . For mappings that are isotopic to periodic self-homeomorphisms the infimum is attained. More precisely, the following theorem holds.

THEOREM 2.5 (see [5]). *A self-homeomorphism φ of a surface S is (free) isotopic to a periodic self-homeomorphism iff there is a conformal structure w on S and a self-homeomorphism $\tilde{\varphi}$ of S such that $w \circ \tilde{\varphi} \circ w^{-1}$ is conformal (thus $K(w \circ \tilde{\varphi} \circ w^{-1}) = 0$). Moreover, if φ is free isotopic to a periodic self-homeomorphism then there is a φ -minimal conformal structure of first kind.*

For self-homeomorphisms of S which are not isotopic to periodic ones the following theorem holds.

THEOREM 2.6 (see [5]). *A conformal structure w of second kind on a surface S cannot be φ -minimal for a self-homeomorphism φ of S that is not free isotopic to a periodic one. Moreover, there exists a conformal structure w_1 of first kind and a self-homeomorphism $\tilde{\varphi}$ of S which is isotopic to φ and such that $K(w_1 \circ \tilde{\varphi} \circ w_1^{-1}) < K(w \circ \varphi \circ w^{-1})$.*

In the light of the two theorems it is sufficient to consider the extremal problem only for conformal structures of first kind. Moreover, we may fix a reference conformal structure of first kind and replace S by the obtained Riemann surface X . Composing the mappings w in (2.32) with the inverse of the reference conformal structure we may consider conformal structures on X rather than on S . Assume that X is of genus g with m punctures and $3g - 3 + m > 0$. (We require that the

universal covering of X is \mathbb{C}_+ and, in case of genus 0, that the number of punctures is at least 4, to avoid trivial cases.)

The Teichmüller space of a Riemann surface $\mathcal{T}(X)$ of first kind can equivalently be described as follows. Consider the set of *all* homeomorphisms of X onto another Riemann surface Y of first kind (instead of *quasiconformal* homeomorphisms). Call two homeomorphisms $w_j : X \rightarrow Y_j$, $j = 1, 2$, pre-Teichmüller equivalent if there is a conformal mapping $c : Y_1 \rightarrow Y_2$ such that $w_1 \circ c \circ w_2^{-1}$ is isotopic to the identity. The equivalence classes for this relation are the same as the Teichmüller classes. Indeed, if the Riemann surfaces X and Y are of first kind and w is an arbitrary homeomorphism from X onto Y then w can be extended to a homeomorphism between closed Riemann surfaces. The extended homeomorphism can be uniformly approximated by smooth, and thus, quasiconformal homeomorphisms. This implies also that the following distance between Teichmüller classes $[w_1]$ and $[w_2]$ is the same as the Teichmüller distance:

$$d([w_1], [w_2]) \stackrel{\text{def}}{=} \inf \left\{ \frac{1}{2} \log K(v_2 \circ v_1^{-1}) : v_j \text{ is pre-Teichmüller equivalent to } w_j, j = 1, 2 \right\}.$$

Notice that for a Riemann surface X of first kind and for homeomorphisms $w_1 : X \rightarrow X_1$, $w_2 : X \rightarrow X_2$, the Teichmüller distance $d_{\mathcal{T}}([w_1], [w_2])$ can also be written as follows:

$$(2.32) \quad d_{\mathcal{T}}([w_1], [w_2]) = \inf \left\{ \frac{1}{2} \log K(g) : g : X_1 \rightarrow X_2 \text{ a surjective homeomorphism} \right. \\ \left. \text{which is isotopic to } w_2 \circ w_1^{-1} \text{ through such homeomorphisms} \right\}.$$

Let X be a Riemann surface of first kind and let φ be a self-homeomorphism of X . Denote by φ^* the modular transformation induced by (the mapping class of) φ on $\mathcal{T}(g, m)$. Put

$$(2.33) \quad L(\varphi^*) \stackrel{\text{def}}{=} \inf_{\tau \in \mathcal{T}(g, m)} d_{\mathcal{T}}(\tau, \varphi^*(\tau)).$$

The quantity $L(\varphi^*)$ is called the translation length of φ^* . Write $\tau = [w]$,. Then $\varphi^*([w]) = [w \circ \varphi]$, and by (2.31) and (2.32)

$$(2.34) \quad \frac{1}{2} \log I(\varphi) = \inf_{\tau \in \mathcal{T}(g, m)} d_{\mathcal{T}}(\tau, \varphi^*(\tau)).$$

Bers uses the following terminology in analogy to the classification of elements of $\text{PSL}(2, \mathbb{Z})$. Consider a modular transformation φ^* , i.e. an element of the modular group $\text{Mod}(g, m)$. A point $\tau \in \mathcal{T}(g, m)$ is called φ^* -minimal, if $d_{\mathcal{T}}(\tau, \varphi^*(\tau)) = L(\varphi^*)$. A modular transformation $\varphi^* \in \text{Mod}(g, m)$ is elliptic, if it has a fixed point in $\mathcal{T}(g, m)$, parabolic, if it has no fixed point but $L(\varphi^*) = 0$, hyperbolic, if $L(\varphi^*) > 0$ and $L(\varphi^*)$ is attained, and pseudohyperbolic, if $L(\varphi^*) > 0$ but $d_{\mathcal{T}}(\tau, \varphi^*(\tau)) > L(\varphi^*)$ for all $\tau \in \mathcal{T}(g, m)$.

A conformal structure w is φ -minimal for a self-homeomorphism φ of X iff $[w]$ is φ^* -minimal.

THEOREM 2.7 (see [5]). *An element $\varphi^* \in \text{Mod}(g, m)$ is elliptic iff it is periodic. This happens iff the absolutely extremal map in the isotopy class of self-homeomorphisms is conformal.*

The following theorem is a reformulation of Thurston's result.

THEOREM 2.8 (see [5]). *For an irreducible self-homeomorphism φ of a Riemann surface X of first kind the modular transformation φ^* of φ is either elliptic or hyperbolic.*

COROLLARY 2.2. *An irreducible self-homeomorphism φ of a Riemann surface X of first kind leads to an absolutely extremal self-homeomorphism $\tilde{\varphi}$ of a Riemann surface Y by isotopy and conjugation with a homeomorphism. For the quasiconformal dilatation $K(\tilde{\varphi})$ of the absolutely extremal mapping $\tilde{\varphi}$ we have*

$$(2.35) \quad \frac{1}{2} \log K(\tilde{\varphi}) = \frac{1}{2} \log I(\varphi) = L(\varphi^*).$$

The following theorem characterizes the absolutely extremal maps with hyperbolic modular transformation in terms of Teichmüller mappings and quadratic differentials.

THEOREM 2.9 (see [5]). *Let X be a Riemann surface of genus g with m punctures, $3g - 3 + m > 0$. A mapping $\varphi : X \rightarrow X$ is absolutely extremal, iff it is either conformal or a Teichmüller mapping satisfying the following two equivalent conditions*

- (i) *the mapping $\varphi \circ \varphi$ is also a Teichmüller mapping with $K(\varphi \circ \varphi) = (K(\varphi))^2$,*
- (ii) *the initial and terminal quadratic differentials of φ coincide.*

Hence, for each absolutely extremal mapping φ with hyperbolic modular transformation there is a quadratic differential ϕ such that φ maps singular points to singular points, it maps the leaves of the horizontal foliation to leaves of the horizontal foliation and leaves of the vertical foliation to leaves of the vertical foliation. These two foliations are transversal outside the common singularity set and are measured foliations by using the metric $ds = |\phi|^{\frac{1}{2}}$ to measure the distance between leaves. The mapping φ decreases the distance between horizontal trajectories by the factor $K^{-\frac{1}{2}}$ and increases the distance between vertical trajectories by the factor $K^{\frac{1}{2}}$. Here K is the quasiconformal dilatation of φ . These properties characterize the pseudo-Anosov mappings (see [37], [12]). The mappings φ with hyperbolic modular transformations φ^* are exactly those which are isotopic to pseudo-Anosov maps. Moreover, there is a unique pseudo-Anosov map in the isotopy class of such φ and it is equal to the absolutely extremal map in the class.

An isometric image of $(-1, 1)$ with metric $\frac{dx}{1-x^2}$ in the Teichmüller space $\mathcal{T}(g, m)$ with Teichmüller metric is called a geodesic line. We will also need the following theorem.

THEOREM 2.10 (see [5]). *If an element $\varphi^* \in \text{Mod}(g, m)$ is of infinite order (i.e. not periodic), then an element $\tau \in \mathcal{T}(g, m)$ is φ^* -minimal iff φ^* leaves a geodesic line in $\mathcal{T}(g, m)$ through τ invariant. The geodesic line is unique.*

Self-homeomorphisms of Riemann surfaces with parabolic or pseudohyperbolic modular transformation are reducible. Notice that self-homeomorphisms with elliptic modular transformation also may be reducible but those with hyperbolic modular transformation are irreducible (see Theorem 2.11(iv) below).

Following Bers [5] we consider now the extremal problem for the quasiconformal dilatation in the case of reducible self-homeomorphisms of Riemann surfaces.

Suppose again X is a Riemann surface of first kind with $3g - 3 + m > 0$ and a self-homeomorphism $\varphi : X \rightarrow X$ is reduced by a non-empty admissible system of curves $\{C_1, \dots, C_\alpha\}$. φ is called maximally reduced by this system if there is no admissible system of more than α curves which reduces a mapping which is isotopic to φ . φ is called completely reduced by this system of curves if for each connected component X_j of the complement $X \setminus \bigcup_{\ell=1}^{\alpha} C_\ell$ and the smallest positive integer N_j , for which $\varphi^{N_j}(X_j) = X_j$, the map $\varphi^{N_j} \mid X_j$ is irreducible.

LEMMA 2.3 ([5]). *A reducible mapping is isotopic to a maximally reduced mapping.*

LEMMA 2.4 ([5]). *If φ is maximally reduced by a system of curves $\{C_1, \dots, C_\alpha\}$ it is completely reduced by this system.*

Let φ be a self-homeomorphism of a Riemann surface X of first kind, $3g - 3 + m > 0$, which is completely reduced by a non-empty admissible system $\{C_1, \dots, C_\alpha\}$ of curves. If φ is not periodic then an absolutely extremal self-mapping of a Riemann surface related to φ by isotopy and conjugation does not exist (see Theorem 2.11 below). But there exists an absolutely extremal self-mapping ψ of a nodal Riemann surface Y . The nodal Riemann surface Y is the image of X by a continuous mapping w which collapses each curve C_j to a point and is a homeomorphism on $X \setminus \bigcup_1^{\alpha} C_j$. The absolutely extremal mapping ψ is related to φ by isotopy on X and semi-conjugation with w . The nodal Riemann surface Y and the absolutely extremal mapping ψ on it can be regarded as a “limit” of a sequence of non-singular Riemann surfaces X_j and self-homeomorphisms φ_j of X_j . For the X_j we have $X_j = w_j(X)$ for a quasiconformal complex structure w_j on X . The φ_j are related to φ by isotopy and conjugation: $w_j^{-1} \circ \varphi_j \circ w_j$ is isotopic to φ on X . The quasiconformal dilatations $K(\varphi_j)$ converge to the infimum $\inf \{K(w \circ \tilde{\varphi} \circ w^{-1}) : w \in QC(X), \tilde{\varphi} \text{ isotopic to } \varphi\} = e^{2L(\varphi^*)}$ and are strictly larger than the infimum.

We will need the details later and give them here. A nodal Riemann surface X (or Riemann surface with nodes) is a one-dimensional complex space, each point of which has a neighbourhood which is either biholomorphic to the unit disc \mathbb{D} in the complex plane or to the set $\{z = (z_1, z_2) \in \mathbb{D}^2 : z_1 z_2 = 0\}$. (A mapping on the latter set is holomorphic if its restriction to either of the sets, $\{(z_1, 0) : z_1 \in \mathbb{D}\}$, and $\{(0, z_2) : z_2 \in \mathbb{D}\}$, is holomorphic.) In the second case the point is called a node. We assume that X is connected and has finitely many nodes. Let \mathcal{N} be the set of nodes. The connected components of $X \setminus \mathcal{N}$ are called the parts of the nodal Riemann surface. We do not require that the set \mathcal{N} of nodes is non-empty. If it is empty we also call the Riemann surface non-singular. Thus, non-singular Riemann surfaces are particular cases of nodal Riemann surfaces.

We will say that a non-singular Riemann surface is of finite type and stable if it has no boundary continuum, has genus g and m punctures with $2g - 2 + m > 0$ (hence the universal covering is \mathbb{C}_+). A connected nodal Riemann surface with finitely many parts, each of which is a stable Riemann surface of finite type, is called of finite type and stable.

Let X and Y be stable nodal Riemann surfaces of finite type. A surjective homeomorphism $\varphi : X \rightarrow Y$ is orientation preserving if its restriction to each part

of X is so. Notice that φ defines a bijection between the parts of X and the parts of Y . The quasiconformal dilatation of φ is defined as

$$(2.36) \quad K(\varphi) = \max_{X_j} K(\varphi | X_j),$$

where X_j runs over all parts of X .

Let φ be an orientation preserving self-homeomorphism of a stable nodal Riemann surface X of first kind. The mapping φ permutes the parts of X along cycles. Let X_0 be a part of X and let n be the smallest number for which $\varphi^n(X_0) = X_0$. Put $X_j = \varphi^j(X_0)$ for $j = 1, \dots, n-1$, and call $(X_0, X_1, \dots, X_{n-1})$ a φ -cycle of length n . The mapping φ is called absolutely extremal if for any φ -cycle $(X_0, X_1, \dots, X_{n-1})$ the restriction $\varphi|_{X_0 \cup \dots \cup X_{n-1}}$ to the Riemann surface $X_0 \cup \dots \cup X_{n-1}$ (which is not connected if $n > 1$) is absolutely extremal. In other words the following holds. Let $w : X \rightarrow Y$ be a homeomorphism onto another nodal Riemann surface Y (considered as conformal structure on X). Let $\hat{\varphi}$ be a self-homeomorphism of X which is isotopic to φ . Let (X_0, \dots, X_{n-1}) be a cycle of parts for φ . Then

$$(2.37) \quad \max_{0 \leq j \leq n-1} K(\varphi | X_j) \leq \max_{0 \leq j \leq n-1} K(w \circ \hat{\varphi} \circ w^{-1} | w(X_j)).$$

Notice that if φ fixes all parts of X then (2.37) is equivalent to the condition that $\varphi | X_j$ is an absolute extremal self-homeomorphism of X_j for each part X_j of X . In this case Theorem 2.8 applied to the parts gives a description of the absolutely extremal self-homeomorphisms.

It will be convenient to have in mind the following lemma which describes the absolutely extremal self-homeomorphisms on the cycles of parts of the nodal surface X .

LEMMA 2.5. *Let $X_0, X_1, \dots, X_{n-1}, X_n \stackrel{\text{def}}{=} X_0$ be non-singular stable Riemann surfaces of finite type. Suppose φ is a self-homeomorphism of $X_0 \cup \dots \cup X_{n-1}$ which permutes the X_j along the n -cycle*

$$X_0 \xrightarrow{\varphi} X_1 \xrightarrow{\varphi} \dots \xrightarrow{\varphi} X_n \stackrel{\text{def}}{=} X_0.$$

Then φ is absolutely extremal iff the following two conditions hold.

- (1) *The mapping $F \stackrel{\text{def}}{=} \varphi^n|_{X_0}$ is absolutely extremal.*
- (2) *One of the two situations occurs.*
 - (2a) *The mapping F is conformal and all φ^j are conformal.*
 - (2b) *The mapping F is pseudo-Anosov and the Teichmüller classes $[\varphi^j] \in \mathcal{T}(X_0)$, $j = 1, \dots, n-1$, have the following property. Let γ_F be the unique geodesic in the Teichmüller metric on the Teichmüller space $\mathcal{T}(X_0)$ which is invariant under the modular transformation F^* of F . The $[\varphi^j]$ divide the bounded segment on the geodesics with endpoints $[id]$ and $[F]$ into n segments, each of which has length equal to*

$$\frac{1}{2n} d_{\mathcal{T}(X_0)}([id], [F]) = \frac{1}{2n} \log K(F),$$

and the mapping $\varphi : \varphi^j(X_0) \rightarrow \varphi^{j+1}(X_0)$ is a Teichmüller map for each j , $j = 1, \dots, n-1$.

Hence, if φ is absolutely extremal, then

$$\frac{1}{2} \log K(\varphi) = \frac{1}{2} \frac{1}{n} \log K(\varphi^n | X_0) = \frac{1}{2} \frac{1}{n} \log(I(\varphi^n | X_0)).$$

Proof. We start with the following observations. Let $\hat{\varphi}$ be any self-homeomorphism of $X_0 \cup \dots \cup X_{n-1}$ which is isotopic to φ and let $w : \bigcup_{j=0}^{n-1} X_j \rightarrow \bigcup_{j=0}^{n-1} Y_j = w(\bigcup_{j=0}^{n-1} X_j)$ be a conformal structure on $\bigcup_{j=0}^{n-1} X_j$. Define $\psi = w \circ \hat{\varphi} \circ w^{-1}$ on $Y = w(X)$. Put $\hat{\varphi}_j = \hat{\varphi}|_{X_j}$, $\psi_j = \psi|_{Y_j}$ for $j = 0, \dots, n-1$, and $w_j = w|_{X_j}$, $j = 1, \dots, n-1$, $X_n = X_0$, and $w_n = w|_{X_n} = w_0$. The diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{\hat{\varphi}} & X_1 & \xrightarrow{\hat{\varphi}} & \dots & \xrightarrow{\hat{\varphi}} & X_{n-1} & \xrightarrow{\hat{\varphi}} & X_n = X_0 \\ \downarrow w_0 & & \downarrow w_1 & & & & \downarrow w_{n-1} & & \downarrow w_n = w_0 \\ Y_0 & \xrightarrow{\psi} & Y_1 & \xrightarrow{\psi} & \dots & \xrightarrow{\psi} & Y_{n-1} & \xrightarrow{\psi} & Y_n = Y_0 \end{array}$$

shows that

$$(2.38) \quad \psi_j = w_{j+1} \circ \hat{\varphi}_j \circ w_j^{-1}, \quad j = 0, \dots, n-1,$$

$$(2.39) \quad \psi^j|_{Y_0} = w_j \circ \hat{\varphi}^j \circ w_0^{-1}, \quad j = 1, \dots, n.$$

Note that $\psi^n|_{Y_0} = w_0 \circ \hat{\varphi}^n \circ w_0^{-1}$. We have the following inequality for the Teichmüller distances between the classes $[\psi^j|_{Y_0}] \in \mathcal{T}(Y_0)$:

$$(2.40) \quad \sum_{j=0}^{n-1} d_{\mathcal{T}(Y_0)}([\psi^j|_{Y_0}], [\psi^{j+1}|_{Y_0}]) \geq d_{\mathcal{T}(Y_0)}([\text{id}], [\psi^n|_{Y_0}]).$$

The right hand side of equation (2.40) equals

$$(2.41) \quad \inf \left\{ \frac{1}{2} \log K(v_2 \circ v_1^{-1}) : v_2 \in [\psi^n|_{Y_0}], v_1 \in [\text{id}|_{Y_0}] \right\} = \inf \left\{ \frac{1}{2} \log K(v_2) : v_2 \in [\psi^n|_{Y_0}] \right\}.$$

Since $\psi^n = w_0 \circ \hat{\varphi}^n \circ w_0^{-1}$ and $\hat{\varphi}^n$ is isotopic to $\varphi^n = F$, the right hand side of (2.41) equals

$$(2.42) \quad \inf \left\{ \frac{1}{2} \log K(v_2) : v_2 \in [w_0 \circ F \circ w_0^{-1}] \right\}.$$

This expression is not smaller than

$$(2.43) \quad \inf \left\{ \frac{1}{2} \log K(\hat{w}_0 \circ \hat{F} \circ \hat{w}_0^{-1}) : \hat{w}_0 : X_0 \rightarrow Y_0 \text{ is a homeomorphism and } \hat{F} \text{ is isotopic to } F \right\} = \frac{1}{2} \log I(F).$$

The left hand side of (2.40) does not exceed

$$\begin{aligned} & n \cdot \max_{0 \leq j \leq n-1} d_{\mathcal{T}(Y_0)}([\psi^j|_{Y_0}], [\psi^{j+1}|_{Y_0}]) = \\ & n \cdot \max_{0 \leq j \leq n-1} \frac{1}{2} \inf \left\{ \log K(\tilde{v}_2 \circ \tilde{v}_1^{-1}) : \tilde{v}_2 \in [\psi^{j+1}|_{Y_0}], \tilde{v}_1 \in [\psi^j|_{Y_0}] \right\} \leq \\ & n \cdot \max_{0 \leq j \leq n-1} \frac{1}{2} \log K(\psi^{j+1} \circ \psi^{-j}|_{Y_0}) = \end{aligned}$$

$$(2.44) \quad n \cdot \max_{0 \leq j \leq n-1} \frac{1}{2} \log K(w \circ \hat{\varphi} \circ w^{-1} | Y_j).$$

We used (2.39).

Hence the last expression $n \cdot \max_{0 \leq j \leq n-1} \frac{1}{2} \log K(w \circ \hat{\varphi} \circ w^{-1} | Y_j)$ in (2.44) is not smaller than $\frac{1}{2} \log I(F)$. It is equal to this quantity iff each term on the left hand side of (2.40) is equal to $\frac{1}{2n} \log I(F)$, otherwise the left hand side of (2.40) is strictly bigger than $\frac{1}{2} \log I(F)$.

These observation imply the lemma. Indeed, suppose $\varphi^n | X_0$ is irreducible and has hyperbolic modular transformation. Then the infimum of the expression $n \cdot \max_{0 \leq j \leq n-1} \frac{1}{2} \log K(w \circ \hat{\varphi} \circ w^{-1} | Y_j)$ in (2.44) is attained if and only if $F = \varphi^n | X_0$ is pseudo-Anosov (so that $K(F) = I(F)$) and condition (2b) of the statement of the lemma holds.

Suppose $\varphi^n | X_0$ has elliptic modular transformation. Then the infimum of the last expression in (2.43) is attained if and only if $F = \varphi^n | X_0$ is conformal (so $K(F) = 0$) and $K(\varphi | X_j) = 0$ for all j , $j = 0, \dots, n-1$. Notice that this happens e.g. if $X_0 = \dots = X_{n-1}$, $\varphi | X_j = id$ for $j = 0, \dots, n-2$, and $\varphi | X_{n-1} = F$. We proved that the self-homeomorphism φ of $X_0 \cup \dots \cup X_{n-1}$ is absolutely extremal if and only if conditions (1) and (2) of the lemma hold, provided $F = \varphi^n | X_0$ is irreducible.

If $F = \varphi^n | X_0$ is reducible φ cannot be absolutely extremal. This statement is obtained similarly as Theorem 2.6 (see [5]). \square

Note that for given $[F]$ with F pseudo-Anosov the choice of the classes $[\psi^j | X_0]$ with the required property is unique. Hence, the $Y_j = \psi^j(X_0)$ are unique up to conformal mappings and the ψ_j are unique up to composing and precomposing with conformal mappings. The ψ_j are Teichmüller mappings and the terminal quadratic differential of ψ_j coincides with the initial quadratic differential of ψ_{j+1} , $j = 1, \dots, n$ ($\psi_{n+1} \stackrel{\text{def}}{=} \psi_1$).

Now we describe in more detail the solution of the extremal problem (2.31) in the reducible case. Let X be a connected Riemann surface, which is closed or of first kind, with universal covering equal to \mathbb{C}_+ . Let $\mathfrak{m} \in \mathfrak{M}(X)$ be an isotopy class of orientation preserving self-homeomorphisms. Let \mathcal{C} be an admissible system of curves which completely reduces an element φ of \mathfrak{m} . By an isotopy we may assume that \mathcal{C} is real analytic (or even geodesic). Associate to X and the system of curves \mathcal{C} a nodal surface Y and a continuous surjection $w : X \rightarrow Y$. This can be done as follows.

Surround each connected component of \mathcal{C} by an annulus which admits a conformal mapping \mathfrak{c} onto a round annulus $A_r = \{\frac{1}{r} < |z| < r\}$ for some $r > 1$, such that \mathfrak{c} maps the curve to the unit circle. Let $\chi : [1, r] \rightarrow [0, r]$ be a diffeomorphism which is the identity near r and maps 1 to 0.

Put the mapping w equal to the identity outside the union of the annuli surrounding the curves in \mathcal{C} . Take coordinates on each annulus which make it a round annulus A_r for some r and define w in these coordinates as follows. For $|z| \leq 1$ we put $w(z) = (0, \chi(|z|^{-1}) \cdot \frac{z}{|z|})$ and for $|z| \geq 1$ we put $w(z) = (\chi(|z|) \cdot \frac{z}{|z|}, 0)$. Then $Y = w(X)$ is a nodal surface and w is a continuous surjection to Y . The nodes of Y are in one-to-one correspondence with the curves in \mathcal{C} . Denote the set of nodes of Y by \mathcal{N} .

Since φ maps each curve in \mathcal{C} to a curve in \mathcal{C} we may consider the function $w \circ \varphi \circ w^{-1}$ on $Y \setminus \mathcal{N}$. It extends continuously to the nodes. Denote the obtained function on Y by $\hat{\varphi}$ and its isotopy class on Y by $\hat{\mathfrak{m}}$.

Note that the nodal surface Y is defined up to isotopies by the isotopy class of \mathcal{C} (the isotopy of \mathcal{C} is within real analytic systems of curves). The class $\hat{\mathfrak{m}}$ is determined by \mathfrak{m} , and Y , and therefore by \mathfrak{m} , and by the system of curves \mathcal{C} . The conjugacy class $\hat{\mathfrak{m}}$ of $\hat{\mathfrak{m}}$ is defined by the isotopy class of \mathcal{C} and the class $\hat{\mathfrak{m}}$. We call the conjugacy classes $\hat{\mathfrak{m}}_j$ of the restrictions $\hat{\mathfrak{m}}_j$ of the class $\hat{\mathfrak{m}}$ to the cycles of parts of the nodal surface the irreducible components of $\hat{\mathfrak{m}}$ related to the class of \mathcal{C} . Notice that the class $\hat{\mathfrak{m}}$ determines the class $\hat{\mathfrak{m}}$ only modulo products of powers of Dehn twists around curves which are homotopic to curves of the system \mathcal{C} .

Notice also that the isotopy class of a system of curves \mathcal{C} which completely reduces an element of \mathfrak{m} is not uniquely determined by X and \mathfrak{m} , even if we require that the system maximally reduces the homeomorphism. In particular, the type of the nodal surface Y is not uniquely determined. This may occur, for instance for reducible homeomorphisms with elliptic modular transformation.

It is a remarkable fact that the extremal problem for the quasiconformal dilatation has a solution in terms of nodal Riemann surfaces and irreducible parts of mapping classes.

The following theorem is due to Bers.

THEOREM 2.11 ([5]). *Let \mathfrak{m} be a mapping class of orientation preserving self-homeomorphisms of a Riemann surface X of first kind with universal covering \mathbb{C}_+ . Let \mathcal{C} be a system of real analytic admissible curves which completely reduces an element $\varphi \in \mathfrak{m}$. Choose a stable nodal Riemann surface Y of first kind with set of nodes \mathcal{N} and a continuous surjection $w : X \rightarrow Y$ which contracts each curve of \mathcal{C} to a point and whose restriction to the complement of \mathcal{C} is a homeomorphism onto $Y \setminus \mathcal{N}$. Denote by $\hat{\mathfrak{m}}$ the mapping class on Y induced by \mathfrak{m} and w , and by $\hat{\mathfrak{m}}$ its conjugacy class. Then the following holds.*

- (1) *There exists a conformal structure \tilde{w} on Y , $\tilde{w} : Y \rightarrow \tilde{w}(Y) = \tilde{Y}$, and an absolutely extremal self-homeomorphism $\tilde{\varphi}$ of \tilde{Y} , representing the class $\hat{\mathfrak{m}}$.*

The self-homeomorphism $\tilde{\varphi}$ has the following stronger extremal properties.

- (2) *We have*

$$I(\varphi) = e^{2L(\varphi^*)} = K(\tilde{\varphi})$$

for the modular transformation φ^ of φ . Moreover, for any nodal Riemann surface Y_1 , any continuous surjection $w_1 : X \rightarrow Y_1$ such that the preimages of the nodes are disjoint Jordan curves and the restriction of w_1 to the complement of the curves is a homeomorphism, and any self-homeomorphism φ_1 in the class $\hat{\mathfrak{m}}_1$ induced by \mathfrak{m} and w_1 , we have*

$$K(\tilde{\varphi}) \leq K(\varphi_1).$$

- (3) *If \mathfrak{m} is reducible and not periodic then for each non-singular Riemann surface Y_1 , each surjective homeomorphism $w_1 : X \rightarrow Y_1$, and each self-homeomorphism φ_1 of Y_1 such that $w_1^{-1} \circ \varphi_1 \circ w_1 \in \mathfrak{m}$, we have strict inequality*

$$K(\tilde{\varphi}) < K(\varphi_1).$$

- (4) *If \mathfrak{m} is reducible then there exists a sequence Y_j of non-singular Riemann surfaces Y_j , surjective homeomorphisms $w_j : X \rightarrow Y_j$, and self-homeomorphism φ_j of X_j with the following property:*

$$w_j^{-1} \circ \varphi_j \circ w_1 \in \mathfrak{m} \text{ and } K(\varphi_j) \rightarrow K(\tilde{\varphi}).$$

Recall that a self-homeomorphism $\tilde{\varphi}$ of \tilde{Y} is absolutely extremal if the infimum of the quantities (2.36) for self-homeomorphisms of nodal surfaces in the class of $\tilde{\varphi}$ is attained for $\tilde{\varphi}$. The infimum $I(\varphi)$ in the first part of assertion (2) is taken over self-homeomorphisms of the original non-singular Riemann surface in the class of the homeomorphism φ . Statement (2) is strictly stronger than statement (1). In statement (1) the conjugacy class of the nodal surface is fixed. In statement (2) the type of the nodal surface Y_1 is not prescribed. It may be different from the type of Y . The first part of statement (2) concerns the case of non-singular surfaces.

The deepest parts of the theorem are its particular case concerning irreducible homeomorphisms and statement (3).

CHAPTER 3

The entropy of surface homeomorphisms

The topological entropy of continuous surjective mappings of a compact topological space to itself is defined as follows.

Let X be a compact topological space and φ a continuous mapping from X onto itself. Let \mathcal{A} be a collection of open subsets of X which cover X (for short, \mathcal{A} is an open cover of X). For two open covers \mathcal{A} and \mathcal{B} put $\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$. Let $\mathcal{N}(\mathcal{A})$ be the minimal cardinality of a subset \mathcal{A}_1 of \mathcal{A} which is a cover of X . The entropy $h(\varphi, \mathcal{A})$ of φ with respect to \mathcal{A} is defined as

$$(3.1) \quad h(\varphi, \mathcal{A}) \stackrel{\text{def}}{=} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \mathcal{N}(\mathcal{A} \vee \varphi^{-1}(\mathcal{A}) \vee \dots \vee \varphi^{-N}(\mathcal{A})).$$

(Here $\varphi^{-1}(\mathcal{A}) = \{\varphi^{-1}(A) : A \in \mathcal{A}\}$, $\varphi^{-k-1}(\mathcal{A}) = \varphi^{-1}(\varphi^{-k}(\mathcal{A}))$.) The entropy $h(\varphi)$ of φ is defined as $\sup_{\mathcal{A}} h(\varphi, \mathcal{A})$.

The entropy is a conjugacy invariant:

if $\psi : X \rightarrow Y$ is a homeomorphism of topological spaces, then $h(\psi \circ \varphi \circ \psi^{-1}) = h(\varphi)$. Also, the following relation holds: for any non-zero integral number n , $h(\varphi^n) = |n| h(\varphi)$.

For more details see [1].

We will be concerned with the topological entropy of self-homeomorphisms of compact surfaces (with or without boundary). Let X be a compact surface (with or without boundary) with a finite set E_n of distinguished points.

Let $\mathfrak{m} \in \mathfrak{M}(X; \partial X, E_n)$ (or $\mathfrak{m} \in \mathfrak{M}(X; \emptyset, E_n)$) be a mapping class. The entropy of the mapping class \mathfrak{m} is defined as follows

$$(3.2) \quad h(\mathfrak{m}) = \inf \{h(\varphi) : \varphi \in \mathfrak{m}\}.$$

Since the topological entropy of a homeomorphism is invariant under conjugation the following holds:

$$(3.3) \quad h(\mathfrak{m}) = h(\hat{\mathfrak{m}}) = \inf \{h(\psi \circ \varphi \circ \psi^{-1}) : \varphi \in \mathfrak{m}, \psi : X \rightarrow Y \text{ a homeomorphism}\}.$$

For a braid $b \in \mathcal{B}_n$ with base point E_n we define the entropy $h(b)$ as the entropy of the mapping class $\mathfrak{m}_b = \Theta_n(b) \in \mathfrak{M}(\bar{D}; \partial D, E_n)$ corresponding to b :

$$h(b) \stackrel{\text{def}}{=} h(\mathfrak{m}_b).$$

By (3.3) the entropy $h(b)$ does not depend on the choice of the base point E_n and is a conjugacy invariant. Hence,

$$h(b) = h(\hat{b}) = h(\widehat{\mathfrak{m}_b}) \stackrel{\text{def}}{=} \inf \{h(\varphi) : \varphi \in \widehat{\mathfrak{m}_b}\}.$$

The entropy of surface homeomorphisms was studied first in exposé 10 of the Asterisque volume dedicated to Thurston's work [12]. Fathi and Shub proved the following theorem.

THEOREM 3.1 ([12]). *Let X be a closed connected Riemann surface of genus $g \geq 2$. For each pseudo-Anosov self-homeomorphism $\varphi_0 : X \rightarrow X$ of X the following equality for the entropy holds:*

$$h(\varphi_0) = h(\widehat{\mathfrak{m}_{\varphi_0}}) = \log K(\varphi_0)^{\frac{1}{2}}$$

$$= \inf \{h(\varphi_1) : \varphi_1 \text{ a self-homeomorphism of } X \text{ which is isotopic to } \varphi_0\}.$$

Here $K(\varphi_0)$ is the quasiconformal dilatation of φ_0 , \mathfrak{m}_{φ_0} is the mapping class of φ_0 and $\widehat{\mathfrak{m}_{\varphi_0}}$ is its conjugacy class.

Ahlfors' trick gives the analog of this theorem for closed (connected) Riemann surfaces of genus g with m distinguished points such that $3g - 3 + m > 0$.

THEOREM 3.2. *Let X be a (closed connected) Riemann surface of genus g with a set E_m of m distinguished points, $3g - 3 + m > 0$. Let $\varphi_0 \in \text{Hom}(X; \emptyset, E_m)$ be a pseudo-Anosov self-homeomorphism of X with distinguished set of points E_m (by an abuse of language identified with a pseudo-Anosov self-homeomorphism of $X \setminus E_m$). Then*

$$\begin{aligned} h(\varphi_0) &= \frac{1}{2} \log K(\varphi_0) = L(\varphi_0^*) = h(\widehat{\mathfrak{m}_{\varphi_0}}) \\ &= \inf \{h(w \circ \varphi \circ w^{-1}) : w : X \rightarrow Y \text{ a homeomorphism onto} \\ &\quad \text{a Riemann surface } Y \text{ with } m \text{ distinguished points,} \\ &\quad \varphi \text{ isotopic to } \varphi_0 \text{ through homeomorphisms of } X \text{ fixing } E_m\}. \end{aligned}$$

Again $K(\varphi_0)$ is the quasiconformal dilatation of φ_0 . φ_0^* is the modular transformation on the Teichmüller space $\mathcal{T}(g, m)$ induced by φ_0 and $L(\varphi_0^*)$ is its translation length.

Together with Corollary 2.2 we obtain the following statement, which includes homeomorphisms with elliptic or hyperbolic modular transformation.

COROLLARY 3.1. *Let φ be an irreducible self-homeomorphism of a (closed connected) Riemann surface X with a set E_m of m distinguished points, $3g - 3 + m > 0$. Let φ_0 be an absolutely extremal self-homeomorphism of a Riemann surface Y with m distinguished points, which is obtained from φ by isotopy and conjugation. Then*

$$\begin{aligned} h(\varphi_0) &= \frac{1}{2} \log K(\varphi_0) = L(\varphi^*) = h(\widehat{\mathfrak{m}_{\varphi_0}}) \\ &= \inf \{h(\varphi_1) : \varphi_1 \text{ is obtained from } \varphi \text{ by isotopy and conjugation}\}. \end{aligned}$$

Theorem 3.2 seems to be folklore among people in Teichmüller theory, but we do not know any reference. For convenience of the reader we provide the proof here.

Proof of Theorem 3.2. Let X be a connected closed Riemann surface with set of distinguished points $\{z_1, \dots, z_m\}$, $3g - 3 + m > 0$. According to [3] there is a closed Riemann surface \hat{X} of genus at least two which is a simple branched covering of X so that the set of branch points project onto the set $\{z_1, \dots, z_m\}$. The pseudo-Anosov homeomorphism φ_0 lifts to a pseudo-Anosov homeomorphism $\hat{\varphi}_0$ on \hat{X} with $K(\varphi_0) = K(\hat{\varphi}_0)$.

The following lemma holds.

LEMMA 3.1. *Let φ be a self-homeomorphism of a compact smooth surface X . Let $\hat{X} \xrightarrow{p} X$ be a topological simple branched covering of X and let $\hat{\varphi}$ be a lift of φ to a self-homeomorphism of \hat{X} . Then $h(\hat{\varphi}) = h(\varphi)$.*

The lemma implies Theorem 3.2. Indeed, by the lemma

$$\begin{aligned} & \inf \{h(\varphi_1) : \varphi_1 \text{ is a self-homeomorphism of } X \text{ which is isotopic to } \varphi\} \\ &= \inf \{h(\hat{\varphi}_1) : \hat{\varphi}_1 \text{ is the lift of a self-homeomorphism of } X \\ & \quad \text{which is isotopic to } \varphi\} \\ &\geq \inf \{h(\mathcal{F}_1) : \mathcal{F}_1 \text{ is a self-homeomorphism of } \hat{X} \text{ which is isotopic to } \hat{\varphi}\}. \end{aligned}$$

Since $\hat{\varphi}_0$ is pseudo-Anosov on \hat{X} and isotopic to $\hat{\varphi}$, the Fathi-Shub Theorem 3.1 implies that the last infimum equals $h(\hat{\varphi}_0)$. Thus, the last infimum is attained on a homeomorphism of \hat{X} which is a lift. Hence the inequality between the second and the third infimum is an equality too. Since by Lemma 3.1 we have $h(\varphi_0) = h(\hat{\varphi}_0)$, the first infimum equals $h(\varphi_0)$. Also, $K(\varphi_0) = K(\hat{\varphi}_0)$. It remains to use Theorem 3.1, Corollary 2.2, and the invariance of the entropy under conjugation. \square

Proof of the Lemma 3.1. The inequality $h(\varphi) \leq h(\hat{\varphi})$ follows from Theorem 5 [1]. Here is a proof for convenience of the reader. Let \mathcal{A} be an open cover of X . Put $\hat{\mathcal{A}} = p^{-1}(\mathcal{A}) \stackrel{\text{def}}{=} \{p^{-1}(A) : A \in \mathcal{A}\}$. $\hat{\mathcal{A}}$ is an open cover of \hat{X} . Since $p \circ \hat{\varphi} = \varphi \circ p$ it follows that $p^{-1}(\varphi^{-1}(\mathcal{A})) = \hat{\varphi}^{-1}(p^{-1}(\mathcal{A}))$ and $p^{-1}(\mathcal{A} \vee \mathcal{B}) = p^{-1}(\mathcal{A}) \vee p^{-1}(\mathcal{B})$ for two open covers \mathcal{A} and \mathcal{B} of X . Hence

$$\begin{aligned} \mathcal{N}(\hat{\mathcal{A}} \vee \dots \vee \hat{\varphi}^{-N}(\hat{\mathcal{A}})) &= \mathcal{N}(p^{-1}(\mathcal{A}) \vee \dots \vee p^{-1}(\varphi^{-N}(\mathcal{A}))) \\ &= \mathcal{N}(p^{-1}(\mathcal{A} \vee \dots \vee \varphi^{-N}(\mathcal{A}))) \\ &= \mathcal{N}(\mathcal{A} \vee \dots \vee \varphi^{-N}(\mathcal{A})). \end{aligned}$$

Hence

$$h(\varphi) = \sup_{\mathcal{A}} h(\varphi, \mathcal{A}) = \sup_{\substack{\hat{\mathcal{A}} = p^{-1}(\mathcal{A}) \text{ for} \\ \text{a cover } \mathcal{A} \text{ of } X}} h(\hat{\varphi}, \hat{\mathcal{A}}) \leq \sup_{\substack{\hat{\mathcal{A}} \text{ an arbitrary} \\ \text{cover of } X}} h(\hat{\varphi}, \hat{\mathcal{A}}) = h(\hat{\varphi}).$$

Prove the opposite inequality. Let $\hat{\mathcal{A}}$ be an open cover of \hat{X} . Define a refinement $\hat{\mathcal{A}}'$ of $\hat{\mathcal{A}}$ as follows. For each $\hat{x} \in \hat{X}$ we take an open connected neighbourhood $\hat{A}_{\hat{x}} \subset \hat{X}$ of \hat{x} which is contained in a set of the cover $\hat{\mathcal{A}}$ and has the following property. If \hat{x} is not a branch point of p , then $p|_{\hat{A}_{\hat{x}}}$ is a homeomorphism onto its image. If \hat{x} is a branch point of p then $p|_{\hat{A}_{\hat{x}}}$ is a double branched covering of its image with single branch point \hat{x} . Let $\hat{\mathcal{A}}'$ be the collection of the $\hat{A}_{\hat{x}}$.

Define a refinement $\hat{\mathcal{A}}''$ of $\hat{\mathcal{A}}'$ as follows. Let x be any point in X , and let $\hat{x}_1, \dots, \hat{x}_\ell$, $\ell \leq m$, be the preimages of x under p . Here m is the covering degree of p . Let $\hat{A}_{\hat{x}_j}$ be the sets of $\hat{\mathcal{A}}'$ defined for \hat{x}_j , $j = 1, \dots, \ell$. Let A_x be the connected component of $p(\hat{A}_{\hat{x}_1}) \cap \dots \cap p(\hat{A}_{\hat{x}_\ell})$ which contains x . Then A_x is a connected open subset of X which contains x . The sets A_x , $x \in X$, form an open cover \mathcal{A}'' of X . Let $\hat{\mathcal{A}}''$ be the covering of \hat{X} which consists of the collection of connected components of $p^{-1}(A_x)$ for all $x \in X$. $\hat{\mathcal{A}}''$ is refining for $\hat{\mathcal{A}}'$, hence for $\hat{\mathcal{A}}$. By property 10 of [1] we have

$$h(\hat{\varphi}, \hat{\mathcal{A}}) \leq h(\hat{\varphi}, \hat{\mathcal{A}}'').$$

Consider the cover $\mathcal{A}'' \vee \varphi^{-1}(\mathcal{A}'') \vee \dots \vee \varphi^{-N}(\mathcal{A}'')$ of X . Take a minimal subcover. The number of sets in the subcover equals $\mathcal{N}(\mathcal{A}'' \vee \varphi^{-1}(\mathcal{A}'') \vee \dots \vee \varphi^{-N}(\mathcal{A}''))$. The sets of the subcover have the form

$$A^0 \cap \varphi^{-1}(A^1) \cap \dots \cap \varphi^{-N}(A^N) \text{ for some } A^k \in \mathcal{A}'', \quad k = 0, \dots, N.$$

Consider the preimages of these sets under p :

$$p^{-1}(A^0 \cap \dots \cap \varphi^{-N}(A^N)) = p^{-1}(A^0) \cap \dots \cap p^{-1}(\varphi^{-N}(A^N)).$$

For each set A^k consider the connected components \hat{A}_j^k of $p^{-1}(A^k)$, $j = 1, \dots, \ell_k$. Here $\ell_k \leq m$. Since $\varphi \circ p = p \circ \hat{\varphi}$,

$$p^{-1}(\varphi^{-k}(A^k)) = \hat{\varphi}^{-k}(p^{-1}(A^k)) = \bigcup_j \hat{\varphi}^{-k}(\hat{A}_j^k).$$

Hence $p^{-1}(A^0 \cap \dots \cap \varphi^{-N}(A^N))$ is the union of at most m sets of the form

$$(3.4) \quad \hat{A}_{j_0}^0 \cap \hat{\varphi}^{-1}(\hat{A}_{j_1}^1) \cap \dots \cap \hat{\varphi}^{-N}(\hat{A}_{j_N}^N).$$

The sets (3.4) belong to the cover $\hat{\mathcal{A}}'' \vee \hat{\varphi}^{-1}(\hat{\mathcal{A}}'') \vee \dots \vee \hat{\varphi}^{-N}(\hat{\mathcal{A}}'')$. If the sets $A^0 \cap \dots \cap \varphi^{-N}(A^N)$ run over a subcover of $\mathcal{A}'' \vee \varphi^{-1}(\mathcal{A}'') \vee \dots \vee \varphi^{-N}(\mathcal{A}'')$, the sets $\hat{A}_{j_0}^0 \cap \dots \cap \hat{\varphi}^{-N}(\hat{A}_{j_N}^N)$ run over a subcover of $\hat{\mathcal{A}}''$. We obtain

$$(3.5) \quad \mathcal{N}(\hat{\mathcal{A}}'' \vee \hat{\varphi}^{-1}(\hat{\mathcal{A}}'') \vee \dots \vee \hat{\varphi}^{-N}(\hat{\mathcal{A}}'')) \leq m \mathcal{N}(\mathcal{A}'' \vee \varphi^{-1}(\mathcal{A}'') \vee \dots \vee \varphi^{-N}(\mathcal{A}'')).$$

Take the limit of $\frac{1}{N} \log$ of the left-hand side and of the right-hand side. We obtain

$$(3.6) \quad h(\hat{\mathcal{A}}'', \hat{\varphi}) \leq h(\mathcal{A}'', \varphi) \leq h(\varphi).$$

Hence, for each cover $\hat{\mathcal{A}}$ of \hat{X}

$$(3.7) \quad h(\hat{\mathcal{A}}, \hat{\varphi}) \leq h(\hat{\mathcal{A}}'', \hat{\varphi}) \leq h(\varphi),$$

therefore

$$(3.8) \quad h(\hat{\varphi}) \leq h(\varphi).$$

□

We will now prove Theorem 4 stated in the introduction which allows to treat the entropy of mapping classes of Riemann surfaces of second kind.

Proof of Theorem 4. Choose coordinates in a neighbourhood of z_0 on X as follows. Let ϕdz^2 be the quadratic differential on X that corresponds to the pseudo-Anosov mapping φ . Equip a neighbourhood of z_0 with distinguished coordinates ζ for ϕdz^2 . In other words, choose coordinates ζ in which the quadratic differential has the canonical form $\zeta^a d\zeta^2$ for an integral number $a \geq -1$, and the fixed point z_0 of φ corresponds to $\zeta = 0$. For a small number $r > 0$ we denote by Δ the round disc $\{|\zeta| < r\}$ in these coordinates. Put $X^{z_0} = X \setminus \{\zeta \in \Delta : |\zeta| \leq \frac{r}{2}\}$. Then $X^{z_0} \subset X$ is a Riemann surface of second kind which is diffeomorphic to $X \setminus \{z_0\}$. We choose the diffeomorphism $X^{z_0} \rightarrow X \setminus \{z_0\}$ which is the identity on $X \setminus \Delta$ and is defined on $X^{z_0} \cap \Delta$ in coordinates ζ by the mapping ψ^{z_0} ,

$$(3.9) \quad \left\{ \frac{1}{2}r < |\zeta| < r \right\} \ni \zeta \rightarrow \psi^{z_0}(\zeta) = \alpha(|\zeta|) \cdot \frac{\zeta}{|\zeta|} \in \Delta \setminus \{0\}$$

for a smooth strictly increasing function $\alpha : (\frac{1}{2}r, r) \rightarrow (0, r)$ with $\lim_{t \rightarrow \frac{r}{2}} \alpha(t) = 0$ and $\alpha(t) = t$ for t close to r .

LEMMA 3.2. *The mapping $(\psi^{z_0})^{-1} \circ \varphi \circ \psi^{z_0}$ extends to a self-homeomorphism φ^{z_0} of $\overline{X^{z_0}}$.*

LEMMA 3.3. $h(\varphi^{z_0}) = h(\varphi)$.

Notice that the mapping φ^{z_0} is not equal to the identity on ∂X^{z_0} .

Proof of Lemma 3.2. Construct first the homeomorphism $\tilde{\varphi}^{z_0}$. Notice that the distinguished coordinates are unique up to multiplication by an $(a+2)^{\text{nd}}$ root of unity. In other words

$$(3.10) \quad \left(\zeta e^{\frac{2\pi i}{a+2} \cdot \ell} \right)^a d \left(\zeta e^{\frac{2\pi i}{a+2} \cdot \ell} \right)^2 = \zeta^a d\zeta^2$$

for an integer ℓ , and this is the only holomorphic change of coordinates in which the quadratic differential has the canonical form.

The initial quadratic differential of φ coincides with its terminal quadratic differential. Hence in the distinguished coordinates ζ the mapping φ has the form

$$(3.11) \quad \varphi(\zeta) = \mathcal{Z}_a(\zeta) \cdot e^{\frac{2\pi i}{a+2} \cdot \ell} \quad \text{for some integer } \ell,$$

and

$$(3.12) \quad \mathcal{Z}_a(\zeta) = \left(\frac{\zeta^{a+2} + 2k|\zeta|^{a+2} + k^2 \bar{\zeta}^{a+2}}{1-k} \right)^{\frac{1}{a+2}}.$$

Here $k = \frac{K-1}{K+1}$, and $K = K(\varphi)$ is the quasiconformal dilatation of φ . We take the root which is positive on the positive real axis. (See e.g. [5], formula (2.3) and Theorem 6 of [5]).

If $a = 0$ (i.e. z_0 is a regular point of the quadratic differential) then the formula is equivalent either to

$$(3.13) \quad \varphi(\zeta) = K^{\frac{1}{2}} \xi + i K^{-\frac{1}{2}} \eta, \quad \zeta = \xi + i\eta,$$

or to

$$(3.14) \quad \varphi(\zeta) = -K^{\frac{1}{2}} \xi - i K^{-\frac{1}{2}} \eta.$$

Suppose $a \neq 0$. The mapping \mathcal{Z}_a maps a neighbourhood of zero on each sector

$$(3.15) \quad S_j = \left\{ \rho e^{i\theta} : \rho \geq 0, \frac{2\pi j}{a+2} \leq \theta < \frac{2\pi(j+1)}{a+2} \right\}, \quad j = 0, \dots, a+1,$$

to a neighbourhood of zero on the same sector. Conjugate the restriction $\mathcal{Z}_a|_{S_j}$ by the branch of the root $\zeta \rightarrow \zeta^{\frac{2}{a+2}}$ which maps

$$(3.16) \quad S' = \{\zeta = \rho e^{i\theta} : \rho \geq 0, \theta \in [0, \pi)\}$$

onto S_j . The conjugated mapping has the form

$$\zeta = \xi + i\eta \rightarrow K^{\frac{1}{2}} \xi + i K^{-\frac{1}{2}} \eta, \quad \zeta \in S'.$$

Write this map in the form

$$(3.17) \quad \zeta = \xi + i\eta = \rho e^{i\theta} \rightarrow \zeta_1 = K^{\frac{1}{2}} \xi + i K^{-\frac{1}{2}} \eta = \rho_1 e^{i\theta_1}, \quad \rho e^{i\theta} \in S'.$$

Here $\cot \theta = \frac{\eta}{\xi}$ and $\cot \theta_1 = \frac{\eta_1}{\xi_1} = \frac{K^{-\frac{1}{2}} \eta}{K^{\frac{1}{2}} \xi} = K^{-1} \cot \theta$ for $\theta \in (0, \pi)$. Hence the mapping $\theta \rightarrow \theta_1$ defines an orientation preserving self-homeomorphism of $(0, \pi)$ which fixes $\frac{\pi}{2}$ and no other point. Hence, the mapping \mathcal{Z}_a has the form

$$(3.18) \quad \mathcal{Z}_a(\zeta) = \varrho(\zeta) \cdot \mathfrak{h} \left(\frac{\zeta}{|\zeta|} \right), \quad \zeta \in \mathbb{C} \setminus \{0\},$$

where ϱ is a continuous function on $\mathbb{C} \setminus \{0\}$ with $\lim_{\zeta \rightarrow 0} \varrho(\zeta) = 0$, and \mathfrak{h} is a self-homeomorphism of the unit circle which fixes the points $e^{i \frac{\pi}{a+2} j}$, $j = 0, 1, \dots, 2(a+2)-1$, and no other point. Similarly, the same expression is obtained for \mathcal{Z}_a in case $a = 0$.

Show that $(\psi^{z_0})^{-1} \circ \varphi \circ \psi^{z_0}$ extends to a self-homeomorphism of $\overline{X^{z_0}}$. Indeed, note that on $\Delta \cap X^{z_0}$ we have in coordinates ζ

$$(3.19) \quad (\psi^{z_0})^{-1}(\zeta) = \beta(|\zeta|) \cdot \frac{\zeta}{|\zeta|},$$

where β is the inverse of $\alpha : \beta(\alpha(t)) = t$, $t \in (\frac{1}{2}r, r)$. Also, $\varphi(\zeta) = \mathcal{Z}_a(\zeta) \cdot e^{\frac{2\pi i}{a+2} \cdot j}$. Hence, on $X^{z_0} \cap \Delta$

$$(3.20) \quad \begin{aligned} (\psi^{z_0})^{-1} \circ \varphi \circ \psi^{z_0}(\zeta) &= (\psi^{z_0})^{-1} \left(\varrho \left(\alpha(|\zeta|) \cdot \frac{\zeta}{|\zeta|} \right) \cdot \mathfrak{h} \left(\frac{\zeta}{|\zeta|} \right) \cdot e^{\frac{2\pi i}{a+2} \cdot j} \right) \\ &= \beta \left(\left| \varrho \left(\alpha(|\zeta|) \cdot \frac{\zeta}{|\zeta|} \right) \right| \right) \cdot \mathfrak{h} \left(\frac{\zeta}{|\zeta|} \right) \cdot e^{\frac{2\pi i}{a+2} \cdot j}. \end{aligned}$$

Since for $|\zeta| \rightarrow \frac{1}{2}r$ we have $\alpha(|\zeta|) \rightarrow 0$, and hence $\left(\alpha(|\zeta|) \cdot \frac{\zeta}{|\zeta|} \right) \rightarrow 0$, we obtain

$$(3.21) \quad \beta \left(\left| \varrho \left(\alpha(|\zeta|) \cdot \frac{\zeta}{|\zeta|} \right) \right| \right) \xrightarrow{|\zeta| \rightarrow \frac{r}{2}} \frac{r}{2}.$$

Hence, $(\psi^{z_0})^{-1} \circ \varphi \circ \psi^{z_0}$ extends to a self-homeomorphism φ^{z_0} of $\overline{X^{z_0}}$. The restriction of φ^{z_0} to the circle $\{|\zeta| = \frac{r}{2}\}$ is equal to the self-homeomorphism

$$(3.22) \quad \frac{r}{2} e^{i\theta} \rightarrow \frac{r}{2} \mathfrak{h}(e^{i\theta}) \cdot e^{\frac{2\pi i}{a+2} \ell}$$

of the circle.

Proof of Lemma 3.3. The mapping φ of Lemma 3.3 is a pseudo-Anosov mapping of a closed connected Riemann surface X of genus g with a set E of m distinguished points with $3g - 3 + m > 0$.

Step 1. We claim that it is enough to prove a slight modification of the lemma for closed connected Riemann surfaces Y of genus at least two. The modification concerns a pseudo-Anosov self-homeomorphism φ of the Riemann surface Y with two fixed points z_0^1 and z_0^2 , and a Riemann surface $Y^{z_0^1, z_0^2}$ with two boundary components, obtained from Y by removing a round disc in distinguished coordinates around each of the points z_0^1, z_0^2 . It states that for the self-homeomorphism $\varphi^{z_0^1, z_0^2}$ of $\overline{Y^{z_0^1, z_0^2}}$ obtained from φ by making the procedure, described in the beginning of the proof of Theorem 4, near both points z_0^1, z_0^2 , the equality of entropies $h(\varphi^{z_0^1, z_0^2}) = h(\varphi)$ holds.

Indeed, we have to reduce the case $g = 1, m \geq 1$ and $g = 0, m \geq 4$ to the case of the modified lemma. If $g = 1$ then X is a torus with distinguished points, among them z_0 . Consider an unbranched covering X' of X . There are two points z_0^1, z_0^2

on X' over z_0 . Consider the double branched cover X'' of X' with branch locus z_0^1, z_0^2 . Then $g(X'') = 2$ and for each of the z_0^1, z_0^2 a punctured neighbourhood of z_0 is doubly covered by a neighbourhood of the point.

If $g = 0$ then $X = \mathbb{P}^1$ and m is at least 4. Consider the double branched covering X' of \mathbb{P}^1 with branch locus consisting of four points, among them the point z_0 which is fixed by φ . Consider X' as torus with distinguished points, among them the lift z'_0 of z_0 to X' . We reduced this case to the previous one.

In any case there is a branched covering of X by a closed Riemann surface of genus at least 2 and so that z_0 is covered by two points z_0^1, z_0^2 on Y and a punctured neighbourhood of z_0 on X is multiply covered by a punctured neighbourhood of each of the points z_0^1, z_0^2 . The pseudo-Anosov mapping φ lifts to a pseudo-Anosov mapping $\tilde{\varphi}$ on the branched cover with $h(\varphi) = h(\tilde{\varphi})$. There are distinguished coordinates ζ near z_0 , and ζ^j near $z_0^j, j = 1, 2$, which are related by the equality $\zeta = (\zeta^j)^b$ for a positive integer b . Hence, the manifold with boundary $\overline{X^{z_0}}$ lifts to a manifold with boundary $\overline{Y^{z_0^1, z_0^2}}$, and φ^{z_0} lifts to self-homeomorphism $\varphi^{z_0^1, z_0^2}$ of $\overline{Y^{z_0^1, z_0^2}}$. If the modified lemma is proved we obtain $h(\varphi^{z_0^1, z_0^2}) = h(\tilde{\varphi})$. Theorem 5 of [1] then implies $h(\varphi^{z_0}) \leq h(\varphi^{z_0^1, z_0^2}) = h(\tilde{\varphi}) = h(\varphi)$. The opposite inequality $h(\varphi) \leq h(\varphi^{z_0})$ follows from the same theorem.

We prove now Lemma 3.3 for surfaces of genus at least two. (The modified lemma is proved in the same way.)

Step 2. Markov partitions. We will use the theorem of Fathi and Shub (see Theorem 3.1) and some parts of its proof (see [12]). We need the notion of Markov partitions for pseudo-Anosov self-homeomorphisms of closed surfaces of genus at least two. It is given in [12], exposé 10, section 4. Recall the definition.

A partition of a compact topological space X is a collection of relatively closed subsets with non-empty interior which cover X and have pairwise disjoint interior. A sequence of partitions \mathfrak{R}_n is refining if each set of \mathfrak{R}_n is contained in a set of \mathfrak{R}_{n-1} and for any sequence of sets $A_n \in \mathfrak{R}_n, n = 1, 2, \dots$, the intersection $\cap A_n$ is either empty or a point. A Markov partition for a pseudo-Anosov homeomorphism is a partition with additional properties related to the homeomorphism.

Let φ be a pseudo-Anosov self-homeomorphism of a closed Riemann surface X of genus at least two. Let ϕdz^2 be the quadratic differential on X associated to φ .

Consider homeomorphic mappings of the closed square $[0, 1] \times [0, 1] \rightarrow X$ with the following properties. For $t \in (0, 1)$ the image of each closed vertical segment $\{t\} \times [0, 1]$ is contained in a leaf of the vertical foliation for the quadratic differential ϕdz^2 . If $t = 0$ or 1 the open vertical segment $\{t\} \times (0, 1)$ is contained in a leaf of the vertical foliation. The endpoints may or may not be mapped to singular points of the quadratic differential. The symmetric statement is required for the images of the horizontal segments $[0, 1] \times \{t\}$. Such mappings are called good birectangles. (Denote the vertical foliation by \mathcal{F}^s and the horizontal foliation by \mathcal{F}^u and notice that their singularities are exactly the singularities of the quadratic differential. With this notation we get a slightly stronger notion than the definition of good birectangles in [12], 10.4.)

A Markov partition (see [12], 10.4) for the pseudo-Anosov homeomorphism $\varphi : X \rightarrow X$ is a finite collection \mathfrak{R} of good birectangles $\{R_j\}$, which cover X , have

pairwise disjoint interior $\text{Int } R_j$ and satisfy the following two conditions (see figure 3.1):

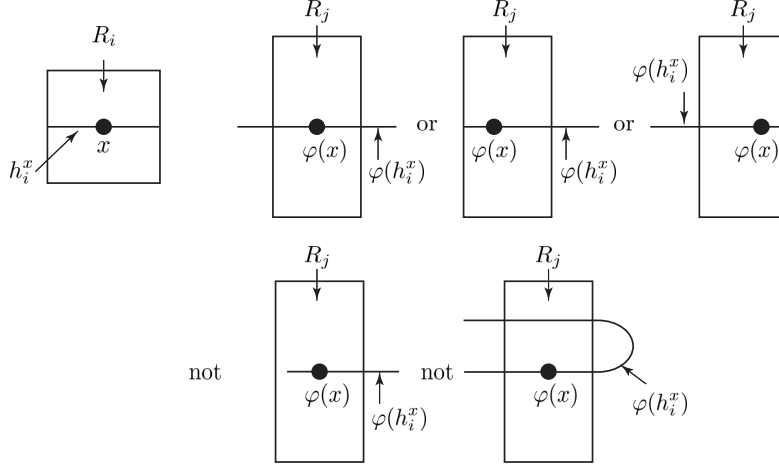


FIGURE 3.1

(1) Suppose $x \in \text{Int } R_i$ and $\varphi(x) \in \text{Int } R_j$. Denote by h_i^x the closed arc in R_i through x which is contained in a horizontal leaf and has endpoints on the vertical sides of R_i . Then $\varphi(h_i^x) \cap R_j$ is a closed arc in R_j containing $\varphi(x)$ with endpoints on the vertical sides of R_j .

(2) Symmetrically, let $v_j^{\varphi(x)}$ be the closed arc in R_j through $\varphi(x)$ which is contained in a vertical leaf and has endpoints on the horizontal sides of R_j . Then $\varphi^{-1}(v_j^{\varphi(x)}) \cap R_i$ is a closed arc in R_i containing x with endpoints on the horizontal sides of R_i .

The existence of Markov partitions (in the slightly stronger sense) for pseudo-Anosov homeomorphisms is proved in [12] (see Proposition 10.17, Lemma 9.4 and Lemma 9.9 of [12]). Fathi and Shub use Markov partitions to relate a finite subshift to a pseudo-Anosov homeomorphism and to show that the entropy of the pseudo-Anosov homeomorphism is equal to the entropy of the subshift (Proposition 10.13 of [12]). This is used to prove Theorem 3.1.

We will need the following details. Put $S_k = \{1, \dots, k\}$ and $\Sigma(k) = \prod_{j=-\infty}^{\infty} S_k^j$.

Here S_k^j is a copy of S_k for each $j \in \mathbb{Z}$. Endow S_k with discrete topology and $\Sigma(k)$ with product topology. Consider a matrix A with entries $a_{ij} = 0$ or 1 for $i, j = 1, \dots, k$. Let Σ_A be the set of infinite sequences $\mathbf{b} = \{b_j\}_{j \in \mathbb{Z}}$ such that

$$(3.23) \quad a_{b_j b_{j+1}} = 1 \quad \text{for each } j \in \mathbb{Z}.$$

Σ_A is a closed subset of $\Sigma(k)$, hence it is compact. The subshift $\sigma_A : \Sigma_A \rightarrow \Sigma_A$ is defined by $\sigma_A(\{b_j\}_{j \in \mathbb{Z}}) = \{b_{j+1}\}_{j \in \mathbb{Z}}$. (By (3.23) the mapping σ_A maps Σ_A to an element of $\Sigma(k)$ which is again in Σ_A .)

Consider a Markov partition $\{R_1, \dots, R_k\}$ of the pseudo-Anosov homeomorphism φ of X . There is a dense open subset $\overset{\circ}{X}$ of X with the following property.

For each $x \in \overset{\circ}{X}$ and each j the point $\varphi^j(x)$ is contained in one of the sets $\text{Int } R_i$. Put $b_j(x) = i$ for $x \in \overset{\circ}{X}$, $j \in \mathbb{Z}$, if $\varphi^j(x) \in \text{Int } R_i$. Hence, for each $x \in \overset{\circ}{X}$ we obtain a point $\mathbf{b}(x) \in \Sigma(k)$. This motivates the following definitions. Let A be the $k \times k$ matrix defined by the condition $a_{ij} = 1$ if $\varphi(\text{Int } R_i) \cap \text{Int } R_j \neq \emptyset$ and $a_{ij} = 0$ otherwise. Associate to φ and the Markov partition the subshift σ_A on $\Sigma_A \subset \Sigma(k)$. Fathi and Shub show that the topological entropy $h(\sigma_A)$ equals the entropy $h(\varphi)$.

The topological entropy of σ_A is expressed in terms of the finite open cover \mathcal{D} by sets C_i , $i = 1, \dots, k$, of Σ_A , where $C_i = \{\mathbf{b} \in \Sigma_A : b_0 = i\}$. The C_i are pairwise disjoint. Then $h(\sigma_A) = h(\sigma_A, \mathcal{D}) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{N}(\mathcal{D} \vee \dots \vee \sigma_A^{-N}(\mathcal{D}))$. Here the number $\mathcal{N}(\mathcal{D} \vee \dots \vee \sigma_A^{-N}(\mathcal{D}))$ of sets in a minimal subcover of $\mathcal{D} \vee \dots \vee \sigma_A^{-N}(\mathcal{D})$ is equal to the number of sets of $\mathcal{D} \vee \dots \vee \sigma_A^{-N}(\mathcal{D})$. By the definition of the matrix A , the number $\mathcal{N}(\mathcal{D} \vee \dots \vee \sigma_A^{-N}(\mathcal{D}))$ is equal to the number $\mathcal{N}(N, \text{Int } \mathfrak{R}^n)$ which we define as the number of non-empty sets of the form

$$(3.24) \quad \text{Int } R_{j_0} \cap \varphi^{-1}(\text{Int } R_{j_1}) \cap \dots \cap \varphi^{-N}(\text{Int } R_{j_N}).$$

We obtain the following.

For each $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that for $N > N(\varepsilon)$ the following inequality holds

$$(3.25) \quad \mathcal{N}(N, \text{Int } \mathfrak{R}^n) \leq (h(\varphi) + \varepsilon)^N.$$

The closures

$$(3.26) \quad \overline{\bigcap_{j=0}^N \varphi^{-j}(\text{Int } R_{j(i)})}$$

of the sets (3.24) cover X . Moreover, by the properties of Markov partitions each of the sets (3.26) is connected (in fact a birectangle).

Let \mathfrak{R} be a Markov partition for the pseudo-Anosov homeomorphism φ of the compact Riemann surface X and let n be a natural number. Consider all non-empty sets of the form $\bigcap_{j=-n}^n \varphi^j(\text{Int } R_{i(j)})$, where φ^j is the j^{th} iterate of φ and $R_{i(j)}$ is an element of \mathfrak{R} for each j . The collection of the closures of such sets is again a Markov partition of X . The sequence \mathfrak{R}^n is a refining sequence of Markov partitions.

Step 3. Refining sequences of partitions of $\overline{X^{z_0}}$. Take n large enough. The partition \mathfrak{R}^n of X induces a partition $\mathfrak{R}^n(z_0)$ of $\overline{X^{z_0}}$ as follows. For each birectangle $R \in \mathfrak{R}^n$ which does not contain z_0 we let its preimage $(\psi^{z_0})^{-1}(R)$ in X^{z_0} be an element of $\mathfrak{R}^n(z_0)$. Let $z_0 \in R$ for a birectangle $R \in \mathfrak{R}^n$. Consider the preimage $(\psi^{z_0})^{-1}(R \setminus \{z_0\})$ in X^{z_0} . Let its closure $\overline{(\psi^{z_0})^{-1}(R \setminus \{z_0\})}$ in $\overline{X^{z_0}}$ be the partition set $R(z_0)$ of $\mathfrak{R}^n(z_0)$ which is related to $\mathfrak{R}^n(z_0)$.

If $z_0 \in R$ then z_0 is a regular point of the quadratic differential ϕdz^2 and $R(z_0)$ differs from $(\psi^{z_0})^{-1}(R \setminus \{z_0\})$ by the circle ∂X^{z_0} .

Suppose z_0 is a singular point of the quadratic differential ϕdz^2 . Then z_0 is a vertex of some birectangles in \mathfrak{R}^n . Let $R \in \mathfrak{R}^n$ contain z_0 as a vertex. (z_0 may be singular or regular.) We may assume (taking n large) that R is contained in a round disc Δ in distinguished coordinates ζ around z_0 . Then R is contained in the closure of a “half-sector” (i.e. of one of the connected components obtained from an S_j by bisecting the angle (see (3.15) for the definition of S_j)). More precisely,

one of the two sides of the rectangle which contain z_0 is the union of z_0 with a half-open arc of a separatrix of the horizontal foliation. The half-open arc is a straight line segment in distinguished coordinates ζ . The other side is the union of z_0 with a half-open arc of a separatrix of the vertical foliation, again a straight line segment in distinguished coordinates ζ . The rectangle does not contain any other arc of a horizontal or vertical leaf that emanates from z_0 . In the considered case $R(z_0)$ differs from $(\psi^{z_0})^{-1}(R \setminus \{z_0\})$ by the arc of the circle ∂X^{z_0} contained in a “half-sector”. (See figures 3.2 a and 3.2 b.)

If z_0 is a regular point and is contained in an open side of a birectangle $R \in \mathfrak{R}^n$ then R is contained in the closure of one of the two sectors of S_1 or S_2 which arise in this case. The set $R(z_0)$ differs from $(\psi^{z_0})^{-1}(R \setminus \{z_0\})$ by a half-circle.

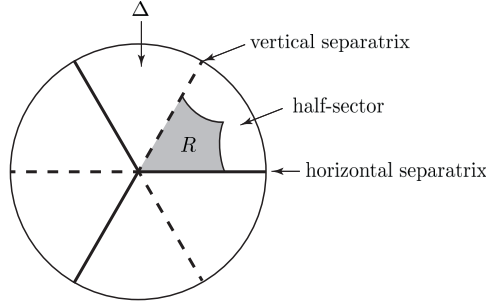


Figure 3.2 a

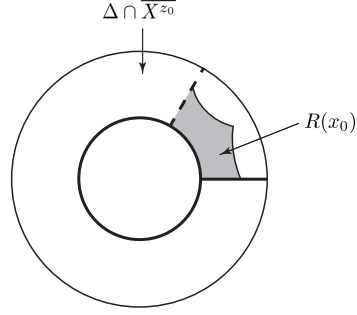


Figure 3.2 b

The partition $\mathfrak{R}^n(z_0)$ is not refining. We will define now for each n a partition $\hat{\mathfrak{R}}^n$ of X which is a refinement of \mathfrak{R}^n so that the induced sequence of partitions $\hat{\mathfrak{R}}^n(z_0)$ of $\overline{X^{z_0}}$ is refining.

Let Δ be again a neighbourhood of z_0 on X which is a round open disc in distinguished coordinates ζ for the quadratic differential ϕdz^2 . Take a natural number n_0 and divide Δ into $j_0 \stackrel{\text{def}}{=} n_0 \cdot 2(a_0 + 2)$ relatively closed subsectors of equal angle by j_0 radii emanating from the center of the disc. Here a_0 is the order of the point z_0 for the quadratic differential. We choose one of the radii along the positive real axis in coordinates ζ . Then all segments of the horizontal and of the vertical separatrices, that emanate from the center of Δ and are contained in Δ , are among the radii. Each closed half-sector (“half” of an $\bar{S}_j \cap \Delta$ (see (3.15))) is the union of some subsectors. Label the subsectors counterclockwise by $\mathfrak{s}_1, \dots, \mathfrak{s}_{j_0}$, so that \mathfrak{s}_1 and \mathfrak{s}_{j_0} are adjacent to the positive real axis in coordinates ζ . (See figure 3.3 a.)

Define the sequence $\hat{\mathfrak{R}}^n$. Let $n_0(n)$, $n = 1, 2, \dots$, be any non-decreasing sequence of natural numbers such that $n_0(n)$ divides $n_0(n+1)$ and $n_0(n) \rightarrow \infty$ for $n \rightarrow \infty$. Let each birectangle $R \in \mathfrak{R}^n$ which does not contain z_0 be an element of $\hat{\mathfrak{R}}^n$. Let $j_0(n) = n_0(n) 2(a_0 + 2)$, and \mathfrak{s}_j , $j = 1, \dots, n_0(n)$, be the sectors of Δ . For a birectangle $R \in \mathfrak{R}^n$ which contains z_0 we let each non-empty set of the form $\hat{R}_j = R \cap \mathfrak{s}_j$ be an element of $\hat{\mathfrak{R}}^n$ (see figure 3.3 a).

The induced partition $\hat{\mathfrak{R}}^n(z_0)$ is obtained in the same way as above. If $\hat{R} \in \hat{\mathfrak{R}}^n$ does not contain z_0 then we let $(\psi^{z_0})^{-1}(\hat{R}) \subset X^{z_0}$ be an element of $\hat{\mathfrak{R}}^n(z_0)$. If

$z_0 \in \hat{R} \in \hat{\mathfrak{R}}^n$, we let $\overline{(\psi^{z_0})^{-1}(\hat{R})}$ be an element of $\hat{\mathfrak{R}}^n(z_0)$. (See figure 3.3.) Since $n_0(n) \rightarrow \infty$ and \mathfrak{R}^n is refining, $\hat{\mathfrak{R}}^n(z_0)$ is a refining sequence of partitions.

LEMMA 3.4. *For each $\varepsilon > 0$ and all natural numbers n and N with $N > N(\varepsilon)$, each set of the form*

$$(3.27) \quad R_0^n \cap \varphi^{-1}(R_1^n) \cap \dots \cap \varphi^{-N}(R_N^n), \quad R_j^n \in \mathfrak{R}^n, \quad j = 0, \dots, N,$$

for which

$$(3.28) \quad \text{Int } R_0^n \cap \varphi^{-1}(\text{Int } R_1^n) \cap \dots \cap \varphi^{-N}(\text{Int } R_N^n)$$

is non-empty, can be covered by no more than $c(n) \cdot N^2$ sets of the form

$$(3.29) \quad \hat{R}_0^n \cap \varphi^{-1}(\hat{R}_1^n) \cap \dots \cap \varphi^{-N}(\hat{R}_N^n), \quad \hat{R}_j^n \in \hat{\mathfrak{R}}^n, \quad j = 0, \dots, N,$$

with

$$(3.30) \quad \text{Int } \hat{R}_0^n \cap \varphi^{-1}(\text{Int } \hat{R}_1^n) \cap \dots \cap \varphi^{-N}(\text{Int } \hat{R}_N^n) \neq \emptyset.$$

End of proof of Lemma 3.3. Define a sequence $\mathcal{A}^n(z_0)$ of open covers of $\overline{X^{z_0}}$ as follows. The elements $A_j^n \in \mathcal{A}^n(z_0)$ are in one-to-one correspondence to the sets $\hat{R}_j^n(z_0) \in \hat{\mathfrak{R}}^n(z_0)$. Moreover, each A_j^n is a neighbourhood of $\hat{R}_j^n(z_0)$ on $\overline{X^{z_0}}$ which is sufficiently close to $\hat{R}_j^n(z_0)$. (See figure 3.3 b.) Since the sequence $\hat{\mathfrak{R}}^n(z_0)$ is refining the sequence $\mathcal{A}^n(z_0)$ can be taken refining.

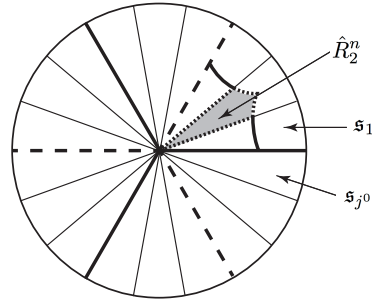


Figure 3.3 a

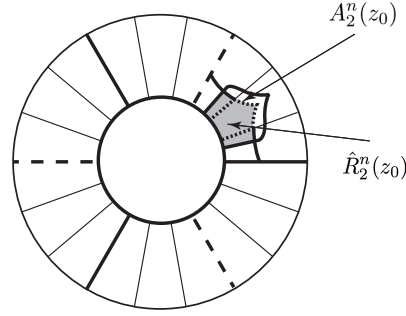


Figure 3.3 b

Estimate the entropy $h(\varphi^{z_0}, \mathcal{A}^n(z_0))$. By (3.25) and Lemma 3.4, for $N > N(\varepsilon)$, the number of sets of the form (3.29) which satisfy (3.30) does not exceed $c(n) \cdot N^2 \cdot (h(\varphi) + \varepsilon)^N$. The elements of \mathfrak{R}^n are in a one-to-one correspondence with the elements of $\hat{\mathfrak{R}}^n(z_0)$ and the latter are in a one-to-one correspondence with the open sets belonging to the cover $\mathcal{A}^n(z_0)$. Take for each set \hat{R}_j^n in (3.29) the set A_j^n corresponding to R_j^n . We obtain a collection of no more than $c(n) \cdot N^2 \cdot (h(\varphi) + \varepsilon)^N$ sets of the form

$$(3.31) \quad A_0^n(z_0) \cap (\varphi^{z_0})^{-1}(A_1^n(z_0)) \cap \dots \cap (\varphi^{z_0})^{-N}(A_N^n(z_0))$$

with $A_j^n(z_0) \in \mathcal{A}^n(z_0)$, which cover $\overline{X^{z_0}}$. In other words,

$$(3.32) \quad \frac{1}{N} \log \mathcal{N}(\mathcal{A}^n(x_0) \vee \dots \vee \varphi^{-N}(\mathcal{A}^n(x_0))) \leq \frac{1}{N} \log(c(n) \cdot N^2 \cdot (h(\varphi) + \varepsilon)^N)$$

for $N > N(\varepsilon)$.

We obtain $h(\varphi^{z_0}, \mathcal{A}^n(z_0)) \leq h(\varphi)$ for each n . Since $\mathcal{A}^n(z_0)$ is refining, $h(\varphi^{z_0}) \leq h(\varphi)$. The opposite inequality is Theorem 5 of [1]. Lemma 3.3 is proved. \square

Proof of Lemma 3.4. Each set $\bar{Q} = R_0^n \cap \varphi^{-1}(R_1^n) \cap \dots \cap \varphi^{-N}(R_N^n)$ of the form (3.27) is connected, and, actually, even a birectangle. Suppose an R_j^n contains z_0 . Taking n large we may assume that each such R_j^n is contained in the neighbourhood Δ of z_0 . Regard the j_0 radii in Δ as $\frac{j_0}{2}$ lines in Δ . Denote the union of these lines by \mathcal{L} . Then $(\text{Int } R_j^n) \setminus \mathcal{L}$ is the union of at most j_0 non-empty connected components (the intersections of $\text{Int } R_j^n$ with some open subsectors). The closures of these sets cover R_j^n .

If R_j^n does not contain a singular point, in particular if $z_0 \in R_j^n$ and z_0 is non-singular, we consider preferred coordinates on R_0^n in which the leaves of the horizontal foliation are segments of horizontal lines and the leaves of the vertical foliation are segments of vertical lines. The coordinates are unique up to a translation and rotation by the angle π .

If R_j^n contains a singular point, then the singular point must be a vertex of R_j^n and we may assume that R_j^n is contained in a small neighbourhood of the singular point. Define distinguished coordinates on a neighbourhood of the singular point and sectors S_i as in (3.15). Then R_j^n is contained in a “half-sector” (“half” of a closed sector \bar{S}_i). Introduce preferred coordinates on the sector \bar{S}_i (and hence on R_j^n) by mapping \bar{S}_i by a branch of the root $\zeta \rightarrow \zeta^{\frac{2}{a_0+2}}$ onto \bar{S}' (see (3.16)).

Notice that for each j the set $\varphi^j(\bar{Q})$ is contained in R_j^n . Consider the mapping φ^{-j} on $\varphi^j(\bar{Q})$ in preferred coordinates on $\varphi^j(\bar{Q})$ and on \bar{Q} . This is a real affine mapping (up to a translation it contracts the horizontal direction by the factor $K^{-\frac{1}{2}j}$ and dilates the vertical direction by the factor $K^{\frac{1}{2}j}$). Hence, φ^{-j} maps each of the maximal (connected) straight line segments contained in $\mathcal{L} \cap \varphi^j(\bar{Q})$ to a straight line segment in \bar{Q} which divides \bar{Q} .

Consider all j for which R_j^n contains z_0 and consider for each such j the preimage $\varphi^{-j}(\varphi^j(\bar{Q}) \cap \mathcal{L})$ in \bar{Q} of the union of the $\frac{j_0}{2}$ lines. The union \mathcal{L}_Q^N over all the j of such preimages consists of no more than $(N+1) \cdot \frac{j_0}{2}$ straight line segments in \bar{Q} each of which divides \bar{Q} . Put $Q = \text{Int } \bar{Q}$. The complement $Q \setminus \mathcal{L}_Q^N$ of the union of line segments is the union of sets of the form in formula (3.30).

LEMMA 3.5. *Let Q be an open rectangle in the plane with sides parallel to the axes. Let \mathcal{L}_Q be the union of at most N_1 lines in the plane. Then the number of connected components of $Q \setminus \mathcal{L}_Q$ does not exceed $2 \cdot (N_1 + 4)(N_1 + 5)$.*

Proof of Lemma 3.5. Associate to each side of Q the line containing it. Let \mathcal{L}'_Q be the union of \mathcal{L}_Q with the four obtained lines. The connected components of $Q \setminus \mathcal{L}_Q$ are bounded polygons. Each polygon is bounded by segments of at least three lines among the collection of the $N_1 + 4$ lines constituting \mathcal{L}'_Q .

Consider first the case when the $N_1 + 4$ lines in \mathcal{L}'_Q are in general position, i.e. through each point of \bar{Q} there are at most two lines in \mathcal{L}'_Q passing through it. Each polygon of $\text{Int } Q \setminus \mathcal{L}_Q$ has at least one vertex. Each vertex is the intersection point of two lines on \mathcal{L}'_Q and each vertex is contained in at most four polygons. Hence the number of polygons is not bigger than 4 times the number of pairs of lines in

\mathcal{L}'_Q . Hence the number of polygons does not exceed

$$4 \cdot \frac{(N_1 + 4)(N_1 + 5)}{2} = 2(N_1 + 4)(N_1 + 5). \quad (\text{See figure 3.4.})$$

Suppose now that the lines are not in general position. Consider a small perturbation of the lines which does not affect the intersection behaviour of the lines outside small neighbourhoods of points through which pass more than two lines. Assume the perturbed lines are in general position, so that the required estimate holds for the number of polygons related to the perturbed lines. Consider an intersection point of $\ell \geq 3$ unperturbed lines. There are 2ℓ unperturbed polygons that intersect a small neighbourhood V of the point. For each of the 2ℓ unperturbed polygons there is a perturbed polygon that intersects the neighbourhood V if the perturbation is small enough. In addition there are small perturbed polygons contained in V whose vertices are intersection points of pairs of perturbed lines. (See figure 3.4.) Hence the number of polygons is bigger in the general position case.

Lemma 3.5 is proved. \square

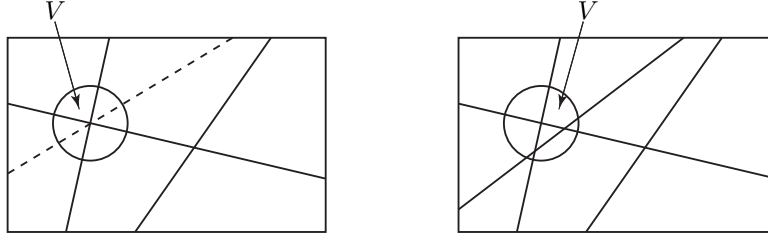


Figure 3.4

In Lemma 3.4 we have $N_1 = (N + 1) \cdot \frac{i_0}{2} = (N + 1) \cdot n_0(n) \cdot (a_0 + 2)$, and the estimate for the number of covering sets is $2((N + 1) \cdot n_0(n)(a_0 + 2) + 4)((N + 1) \cdot n_0(n)(a_0 + 2) + 5)$ which has the desired form for fixed n with a constant $c(n)$ depending only on n (and on X and on the class of the homeomorphism φ).

Lemma 3.4 is proved. \square

End of proof of Theorem 4. Theorem 4 is now a consequence of the following two Lemmas.

LEMMA 3.6. *There exists a self-homeomorphism $\tilde{\varphi}$ of $\{\frac{r}{4} \leq |\zeta| \leq \frac{r}{2}\} \subset \Delta \subset X$ of entropy zero which equals φ^{z_0} on $\{|\zeta| = \frac{r}{2}\}$ and is equal to multiplication by $e^{\frac{2\pi i \ell}{2(a+2)}}$ on $\{|\zeta| = \frac{r}{4}\}$.*

LEMMA 3.7. *Let $\hat{\varphi}$ be a twist on an annulus $\{\frac{r}{8} \leq |\zeta| \leq \frac{r}{4}\}$, in other words, for some real constant μ we have $\hat{\varphi}(\zeta) = \zeta \cdot e^{i\mu \cdot (\log |\zeta| - \log \frac{r}{8})}$. Then $h(\hat{\varphi}) = 0$.*

Indeed, put $\varphi_0 = \varphi^{z_0}$ on $\overline{X^{z_0}}$, put φ_0 equal to the mapping $\tilde{\varphi}$ of Lemma 3.6 on the annulus $\{\frac{r}{4} \leq |\zeta| \leq \frac{r}{2}\}$ in distinguished coordinates around z_0 . Let φ_0 be equal to the homeomorphism $\hat{\varphi}$ of Lemma 3.7 with $\mu = \frac{2\pi \ell}{2(a+2) \cdot \log 2}$ on the

annulus $\{\frac{r}{8} \leq |\zeta| \leq \frac{r}{4}\}$ and equal to the identity on $\{|\zeta| \leq \frac{r}{8}\}$. Then φ_0 is a self-homeomorphism of X that satisfied the requirements of Theorem 4. \square

We will prove now Lemmas 3.6 and 3.7. Recall that in distinguished coordinates ζ on Δ the restriction $\varphi^{z_0} | \partial X^{z_0}$ has the form $\frac{r}{2} e^{\frac{2\pi i \ell}{a+2}} \mathfrak{h}\left(\frac{\zeta}{|\zeta|}\right)$ (see (3.22)). Here $\mathfrak{h}\left(\frac{\zeta}{|\zeta|}\right)$ is a self-homeomorphism of the unit circle which fixes the points $e^{\frac{2\pi i}{2(a+2)}j}$, $j = 0, 1, \dots, 2(a+2) - 1$, and no other point. Note that $\frac{r}{2} \mathfrak{h}\left(\frac{\zeta}{|\zeta|}\right)$ maps each arc $\gamma_j = \left\{ \frac{r}{2} e^{i\theta} : \frac{2\pi j}{2(a+2)} \leq \theta \leq \frac{2\pi(j+1)}{2(a+2)} \right\}$, $j = 1, \dots, 2(a+2) - 1$, homeomorphically onto itself.

LEMMA 3.8. *Fix j . Let \mathcal{Z} be a self-homeomorphism of γ_j which fixes the end-points of γ_j pointwise and does not fix any other point. Then there is a self-homeomorphism $\tilde{\mathcal{Z}}$ of the truncated sector $\bar{\Omega}_j = \{\rho e^{i\theta} : \frac{r}{4} \leq \rho \leq \frac{r}{2}, \frac{r}{2} e^{i\theta} \in \gamma_j\}$ which equals the identity on the boundary part $\partial \Omega_j \setminus \text{Int } \gamma_j$ of $\partial \Omega_j$, equals \mathcal{Z} on γ_j , and has entropy zero.*

Proof. Map $\bar{\Omega}_j$ by a diffeomorphism W onto $\bar{Q} \stackrel{\text{def}}{=} [0, 1] \times [0, 1]$, so that γ_j is mapped to $[0, 1] \times \{1\}$. Identify $[0, 1] \times \{1\}$ with $[0, 1]$. Denote by $\alpha_j = (W | \gamma_j) \circ \mathcal{Z} \circ (W | \gamma_j)^{-1}$ the self-homeomorphism of $[0, 1]$ which is obtained by conjugating \mathcal{Z} with $W | \gamma_j$. We will find a suitable extension α of α_j to $[0, 1] \times [0, 1]$ and put $\tilde{\mathcal{Z}} = W^{-1} \circ \alpha \circ W$.

Define the following self-homeomorphism α of $[0, 1] \times [0, 1]$:

$$(3.33) \quad \alpha(x, y) = (y \alpha_j(x) + (1 - y)x, y), \quad (x, y) \in [0, 1] \times [0, 1].$$

The map α is equal to the identity on $\partial Q \setminus ([0, 1] \times \{1\})$, and equals $(\alpha_j(x), 1)$ for $(x, 1) \in [0, 1] \times \{1\}$. To prove that the conjugate $\tilde{\mathcal{Z}} = W^{-1} \circ \alpha \circ W$ of α by the diffeomorphism $\Omega_j \rightarrow \bar{Q}$ has the desired properties it remains to prove the following lemma. The lemma implies that $h(\tilde{\mathcal{Z}}) = 0$ and, hence, proves Lemma 3.8. \square

LEMMA 3.9. *The homeomorphism α has entropy zero.*

Proof. Consider a refining sequence of partitions of $\bar{Q} = [0, 1] \times [0, 1]$ into squares. Take a partition of the sequence. It is a partition into squares \bar{Q}_k by n_0 horizontal lines and n_0 vertical lines for a natural number n_0 . Denote by $Q = \text{Int } \bar{Q}$, $Q_k = \text{Int } \bar{Q}_k$ the open squares, and by $\mathcal{N}(N, \{Q_k\})$ the number of non-empty sets of the form

$$(3.34) \quad Q_0 \cap \alpha^{-1}(Q_1) \cap \dots \cap \alpha^{-N}(Q_N)$$

for the open squares Q_k of the partition. As in the proof of Lemma 3.3 we will prove the inequality

$$(3.35) \quad \mathcal{N}(N, \{Q_k\}) \leq c(n_0) \cdot N^2$$

for a constant $c(n_0)$ depending on n_0 and α but not on N . This shows that $h(\alpha) = 0$. Indeed, associate to the sequence of partitions a refining sequence of open covers of \bar{Q} . Take the partition by n_0 horizontal and n_0 vertical lines. Consider the associated open cover. The estimate (3.35) shows that the entropy of α with respect to this cover does not exceed $\overline{\lim}_{N \rightarrow \infty} \frac{\log(c(n_0) N^2)}{N} = 0$. Since the sequence of covers is refining, we obtain $h(\alpha) = 0$.

We will now prove (3.35). Denote the union of the n_0 horizontal and n_0 vertical lines by \mathcal{L}_{n_0} . The number $\mathcal{N}(N, \{Q_k\})$ is equal to the number of connected components of $Q \setminus \bigcup_{m=0}^N \alpha^{-m}(\mathcal{L}_{n_0})$. The intersection behaviour of the $\alpha^{-m}(\mathcal{L}_{n_0})$ can be described as follows. Since 0 and 1 are the only fixed points of α_j , either $\alpha_j(x) > x$ for all $x \in (0, 1)$ or $\alpha_j(x) < x$ for all $x \in (0, 1)$. Assume the first inequality holds. Put

$$(3.36) \quad \alpha_y(x) = y \alpha_j(x) + (1 - y)x, \quad x \in [0, 1].$$

Then $\alpha(x, y) = (\alpha_y(x), y)$.

The mapping α maps any vertical segment $\{x\} \times [0, 1]$ to the curve

$$(3.37) \quad C_x^1 = \{(\alpha_y(x), y) : y \in [0, 1]\}.$$

The curves C_x^1 and $\{x\} \times [0, 1]$ have one common point $(x, 0)$. We claim that the curve C_x^1 intersects any vertical segment $\{x'\} \times [0, 1]$ at most once. Indeed, suppose $\alpha_y(x) = x'$ for some $x \in (0, 1)$ and $y \in [0, 1]$. If $y' \in [0, 1]$, $y' > y$, then

$$\alpha_{y'}(x) = y' \cdot \alpha_j(x) + (1 - y') \cdot x > y \cdot \alpha_j(x) + (1 - y) \cdot x = \alpha_y(x)$$

since $\alpha_j(x) > x$ for $x \in (0, 1)$. The opposite inequality holds for $y' < y$, $y, y' \in [0, 1]$.

Hence, C_x^1 and the vertical segment $\{x'\} \times [0, 1]$ divide a neighbourhood of their intersection point into four connected components.

Since α_j is increasing on $[0, 1]$, also for the m -th iterate α^m of α the curve

$$(3.38) \quad C_x^m \stackrel{\text{def}}{=} \alpha^m(\{x\} \times [0, 1]) = \{((\alpha_y)^m(x), y) : y \in [0, 1]\}$$

intersects each vertical segment $\{x'\} \times [0, 1]$ at most once. (Here we denoted by $(\alpha_y)^m$ the m -th iterate of α_y .) Indeed, by induction, if for $x \in (0, 1)$, $y, y' \in [0, 1]$ and $y' > y$ we have $(\alpha_{y'})^k(x) > (\alpha_y)^k(x)$, then

$$(\alpha_{y'})^{(k+1)}(x) = \alpha_{y'}((\alpha_{y'})^k(x)) > \alpha_{y'}((\alpha_y)^k(x)) > \alpha_y((\alpha_y)^k(x)).$$

The first inequality uses that $\alpha_{y'}$ is a strictly increasing homeomorphism of $[0, 1]$ and $(\alpha_{y'})^k(x) > (\alpha_y)^k(x)$. The second inequality uses that $x' = \alpha_y^k(x) < 1$ (since $y < 1$), and the fact that $\alpha_{y'}(x') > \alpha_y(x')$. Similarly, for $0 \leq y' < y \leq 1$ we have $(\alpha_{y'})^k(x) < (\alpha_y)^k(x)$, $x \in (0, 1)$, for all natural k . We obtained the following:

each pair of curves C_x^m and $C_{x'}^{m'}$, with $x, x' \in [0, 1]$, and m, m' non-negative integers, intersects at most once and if they intersect they divide a neighbourhood of the intersection point into at most four connected components.

The set $\bigcup_{m=0}^N \alpha^{-m}(\mathcal{L}_{n_0})$ is the union of n_0 horizontal line segments, and $(N + 1) \cdot n_0$ arcs, each being the preimage under some α^m ($0 \leq m \leq N$) of one of the n_0 vertical lines. Take the collection of these $(N + 2) \cdot n_0$ curves and the four line segments constituting the boundary of Q . We have $(N + 2)n_0 + 4$ (connected) curves. Each pair of curves intersects at most once. As in the proof of Lemma 3.3 the connected components of $Q \setminus \bigcup_{m=0}^N \alpha^{-m}(\mathcal{L}_{n_0})$ are generalized polygons whose vertices are intersections of pairs among the $(N + 2)n_0 + 4$ curves and whose sides are segments on some of these curves. By the same arguments as in the proof of Lemma

3.3 the number of generalized polygons does not exceed $4 \cdot \frac{((N+2)n_0+4)((N+2)n_0+5)}{2}$. This proves (3.35). Lemma 3.9 is proved. \square

Proof of Lemma 3.6. Let k be the smallest positive integer for which $k \cdot \ell$ is a multiple of $2(a+2)$. Then φ^{z_0} permutes the γ_j in cycles of length k and $(\varphi^{z_0})^k$ fixes each γ_j . In particular $(\varphi^{z_0})^k \mid \gamma_j$ satisfies the conditions of Lemma 3.8 for each j .

Let $\gamma_{\ell_1} \xrightarrow{\varphi^{z_0}} \gamma_{\ell_2} \longrightarrow \dots \longrightarrow \gamma_{\ell_k} \xrightarrow{\varphi^{z_0}} \gamma_{\ell_1}$ be one of the cycles for φ^{z_0} . For each ℓ_j , $j = 1, \dots, k-1$, take any homeomorphism $\tilde{\varphi}_{\ell_j} : \bar{\Omega}_{\ell_j} \rightarrow \bar{\Omega}_{\ell_{j+1}}$ which equals $\varphi^{z_0} \mid \gamma_{\ell_j}$ on γ_{ℓ_j} and equals multiplication by $e^{\frac{2\pi i \ell}{2(a+2)}}$ on the rest of the boundary of $\bar{\Omega}_{\ell_j}$. Let ψ_{ℓ_1} be the self-homeomorphism of $\bar{\Omega}_{\ell_1}$ obtained by applying Lemma 3.7 to $(\varphi^{z_0})^k \mid \gamma_{\ell_1}$. We put $\tilde{\varphi}_{\ell_k} = \psi_{\ell_1} \circ (\tilde{\varphi}_{\ell_{k-1}} \circ \dots \circ \tilde{\varphi}_{\ell_1})^{-1}$. Then $\tilde{\varphi}_{\ell_k} \circ \dots \circ \tilde{\varphi}_{\ell_2} \circ \tilde{\varphi}_{\ell_1} = \psi_{\ell_1}$ on γ_{ℓ_1} . On each γ_{ℓ_j} we obtain the mapping $\tilde{\varphi}_{\ell_{j-1}} \circ \dots \circ \tilde{\varphi}_{\ell_1} \circ \tilde{\varphi}_{\ell_k} \circ \dots \circ \tilde{\varphi}_{\ell_{j+1}} \circ \tilde{\varphi}_{\ell_j}$ which is conjugate to ψ_{ℓ_1} . Denote this mapping by ψ_{ℓ_j} . By Lemma 3.7 $h(\psi_{\ell_1}) = 0$, hence $h(\psi_{\ell_j}) = 0$ for all j .

Proceed in the same way with all cycles of γ_j under iteration by φ^{z_0} .

The obtained mappings $\tilde{\varphi}_j$ map $\bar{\Omega}_j$ homeomorphically onto $e^{\frac{2\pi i \ell}{2(a+2)}} \bar{\Omega}_j$. They match together to give a well-defined self-homeomorphism $\tilde{\varphi}$ of $\{\frac{r}{4} \leq |\zeta| \leq \frac{r}{2}\}$ which equals φ^{z_0} on $\{|\zeta| \leq \frac{r}{2}\}$ and equals rotation by $e^{\frac{2\pi i \ell}{2(a+2)}}$ on $\{\frac{r}{4} \leq |\zeta| \leq \frac{r}{2}\}$. The ψ_j match together to give a self-homeomorphism ψ of the annulus such that $\tilde{\varphi}^k = \psi$. Since $h(\psi) = 0$ we have $h(\tilde{\varphi}) = 0$. \square

Proof of Lemma 3.7. Denote by A the annulus $A = \{\frac{r}{8} \leq |\zeta| \leq \frac{r}{4}\}$. Take a refining sequence of partitions of A so that each partition \mathfrak{R}_{n_0} is obtained by dividing A by $n_0 - 1$ circles $\text{Circ}_j = \{|\zeta| = \frac{r}{8} \cdot (1 + \frac{j}{n_0})\}$, $j = 1, \dots, n_0 - 1$, and n_0 radii $\text{Rad}_j = \{\rho \cdot e^{\frac{2\pi i}{n_0} j} : \frac{r}{8} \leq \rho \leq \frac{r}{4}\}$, $j = 1, \dots, n_0$. The ℓ -th iterate of $\hat{\varphi}^{-1}$ maps each circle Circ_j onto itself and maps each radius Rad_j onto the curve

$$(3.39) \quad \hat{\varphi}^{-\ell}(\text{Rad}_j) = \left\{ \rho \cdot e^{i \frac{2\pi}{n_0} j + i \cdot \ell \cdot \mu \left(x - \log \frac{r}{8}\right)} : \log \frac{r}{8} \leq x \leq \log \frac{r}{4} \right\}.$$

For each natural N the union of the circles Circ_j , $j = 1, \dots, n_0$, and of the curves $\hat{\varphi}^{-\ell}(\text{Rad}_j)$, $j = 1, \dots, n_0$, $\ell = 0, \dots, N$, defines a partition

$$(3.40) \quad \mathfrak{R}_{n_0} \vee (\hat{\varphi})^{-1}(\mathfrak{R}_{n_0}) \vee \dots \vee (\hat{\varphi})^{-N}(\mathfrak{R}_{n_0})$$

of the annulus into closed sets.

The number of partition sets is estimated as follows. Cut the annulus A along the real axis and map the set $A \setminus \mathbb{R}$ by the logarithm onto the set

$$(3.41) \quad \mathcal{Q} = \left\{ \log \frac{r}{8} \leq \text{Re } \zeta \leq \log \frac{r}{4}, 0 < \text{Im } \zeta < 2\pi \right\}.$$

Denote by $\bar{\mathcal{Q}}$ the closure of \mathcal{Q} . The $n_0 - 1$ circles in A correspond to n_0 closed vertical line segments in $\bar{\mathcal{Q}}$ and the n_0 radii correspond to n_0 horizontal line segments in $\bar{\mathcal{Q}}$. For each $\ell \geq 1$ and each j the set $\hat{\varphi}^{-\ell}(\text{Rad}_j)$ corresponds to at most $c(\mu) \cdot \ell$ closed straight line segments in $\bar{\mathcal{Q}}$. Here $c(\mu)$ is a constant depending only on μ . Indeed, the curve $\hat{\varphi}^{-\ell}(\text{Rad}_j)$ meets \mathbb{R} at most $c(\mu) \cdot \ell$ times. Hence it is divided by \mathbb{R} into at most $c(\mu) \cdot \ell$ connected components. In logarithmic coordinates these connected components are straight line segments. We obtain a partition of $\bar{\mathcal{Q}}$ by the union of the closures of all mentioned line segments.

Consider the connected components of the complement of the line segments in \bar{Q} . The collection of their closures defines a partition of \bar{Q} . The estimate of the number of partition sets of \bar{Q} is now done along the same lines as the proof of Lemma 3.4. We have $n_0 - 1$ closed horizontal line segments and for each ℓ there are at most $c(\mu) \cdot n_0 \cdot \ell$ closed line segments in \bar{Q} corresponding to a set (3.39) for some j . Hence the total number of these line segments does not exceed $2n_0 + c(\mu) \cdot n_0 \cdot \frac{(N+1) \cdot (N+2)}{2}$. As in the proof of Lemma 3.4, these lines define a partition of \bar{Q} into no more than $C(\mu, n_0) \cdot N^4$ polygons. The constant $C(\mu, n_0)$ depends only on μ and n_0 .

The partition of \bar{Q} corresponds to the partition of the annulus into closed sets of the partition (3.40). The number of the partition sets does not exceed $C(\mu, n_0) \cdot N^4$. Associating to the partition of the annulus an open cover we obtain as in the proof of Lemma 3.4 that the entropy of $\hat{\varphi}$ with respect to this cover does not exceed $\lim_{N \rightarrow \infty} \frac{1}{N} \log(C(\mu, n_0) \cdot N^4) = 0$. The estimate is obtained for each element of a refining sequence of open covers of the annulus. Hence $h(\hat{\varphi}) = 0$. \square

The following theorem concerns the slightly more general case when the self-homeomorphism φ of a Riemann surface of first kind is changed without increasing entropy on the union of discs around points of a subset of the set of distinguished points (rather than on a single disc around a fixed distinguished point).

Theorem 3.4. *Let X be a connected closed Riemann surface with a set E of distinguished points. Assume that $X \setminus E$ is hyperbolic. Let φ be a pseudo-Anosov self-homeomorphism which fixes E setwise. Suppose there is a φ -invariant subset $E' \subset E$, and the entropy $h(\varphi)$ of φ is finite.*

Then φ is isotopic through self-homeomorphisms which fix E setwise to a self-homeomorphism φ_0 of the same entropy $h(\varphi_0) = h(\varphi)$ with the following property.

For each $z \in E'$ there is a closed round disc $\bar{\delta}_z$ in distinguished coordinates for the quadratic differential of φ , such that z is the center of δ_z . Moreover, φ_0 maps each δ_z , $z \in E'$, conformally onto another disc of the collection and maps the center of the source disc to the center of the target disc. Moreover, if a φ -cycle of points in E' has length k , then the iterate φ_0^k fixes pointwise the disc δ_z around each point z of the cycle.

The proof of Theorem 3.4 follows the same lines as the proof of Theorem 3.3 and is left of the reader.

We have the following corollaries. We formulate Corollary 3.2 concerning mapping classes of braids separately because it is simple and useful, although it is a particular case of Corollary 3.3.

Corollary 3.2. *Let $b \in \mathcal{B}_n$ be an irreducible braid and let $\mathbf{m}_b = \Theta_n(b) \in \mathfrak{M}(\mathbb{D}; \partial \mathbb{D}, E_n)$ be its mapping class. Then*

$$h(\widehat{\mathbf{m}_b}) = h(\widehat{\mathbf{m}_b \cdot \mathbf{m}_{\Delta_n^{2k}}}) = h(\widehat{\mathbf{m}_{b, \infty}})$$

for each integer k .

Recall that $\mathbf{m}_{b, \infty} = \mathcal{H}_\infty(\mathbf{m}_b)$ (see section 2), Δ_n is the Garside element in \mathcal{B}_n and by $\widehat{\mathbf{m}}$ we denote the conjugacy class of a mapping class \mathbf{m} .

Proof. Identify the set of elements of \mathfrak{m}_b with the set of elements of $\mathfrak{m}'_b \subset \mathfrak{m}_{b,\infty}$ which are equal to the identity outside the unit disc $\overline{\mathbb{D}}$. The entropy of a class is the infimum of entropies of mappings in the class. We obtain the inequality $h(\widehat{\mathfrak{m}}_b) = h(\widehat{\mathfrak{m}'_b}) \geq h(\widehat{\mathfrak{m}_{b,\infty}})$.

On the other hand Theorem 3.3 assigns to each absolutely extremal representative of $\widehat{\mathfrak{m}_{b,\infty}}$ a representative of $\widehat{\mathfrak{m}_{b,\infty}}$ which has the same entropy and equals the identity in a neighbourhood of infinity and, thus represents $\widehat{\mathfrak{m}'_b}$. Hence $h(\widehat{\mathfrak{m}_{b,\infty}}) \geq h(\widehat{\mathfrak{m}'_b}) = h(\widehat{\mathfrak{m}}_b)$. The first equation of the statement of the corollary is Lemma 3.7. \square

Corollary 3.3. *Let X be a bordered Riemann surface, and let E be a finite set of distinguished points in $\text{Int } X$. Let \mathfrak{m} be an irreducible relative isotopy class of mappings, $\mathfrak{m} \in \mathfrak{M}(X; \partial X, E)$. Then the following equalities hold*

$$h(\widehat{\mathfrak{m}}) = h(\widehat{\mathfrak{m} \cdot \mathfrak{m}_D}) = h(\widehat{\mathcal{H}_\zeta \mathfrak{m}}) = h(\widehat{\mathcal{H}_\partial \mathfrak{m}}),$$

where \mathfrak{m}_D is the mapping class of an arbitrary product of powers of Dehn twists about simple closed curves which are free homotopic to boundary curves of X .

Recall that $\mathcal{H}_\partial \mathfrak{m}$ is the mapping class corresponding to \mathfrak{m} on the N -points compactification of $\text{Int } X$ (see section 2), and $\mathcal{H}_\zeta \mathfrak{m}$ is the mapping class corresponding to \mathfrak{m} in $\mathfrak{M}(X^c; \zeta, E)$ for a compact Riemann surface X^c containing X as a subset and a set ζ of points of $X^c \setminus X$ containing exactly one point in each connected component of $X^c \setminus X$ (see section 2).

The proof follows along the same lines as the proof of Corollary 3.2. It relies on Theorem 3.4.

CHAPTER 4

Proof of Theorem 1. The upper bound for the conformal module. The irreducible case.

Let $\hat{b} \in \hat{\mathcal{B}}_n$ be a conjugacy class of braids and let $f : A \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$ be a holomorphic mapping of an annulus into the symmetrized configuration space which represents \hat{b} . We want to give an upper bound for the conformal module $m(A)$.

For a number $\rho > 1$ we denote by Λ_ρ the linear map $z \rightarrow \rho z$ on the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$. The quotient $\mathbb{C}_+/\Lambda_\rho$ is conformally equivalent to an annulus of conformal module $\frac{\pi}{\log \rho}$. Indeed, the curvilinear rectangle $\{re^{i\theta} : 1 \leq r < \rho, 0 < \theta < \pi\}$ is a fundamental polygon. The logarithm maps it to $\{x + iy : 0 \leq x < \log \rho, 0 < y < \pi\}$. Identifying points on the vertical sides with equal y -coordinate we obtain an annulus of conformal module $\frac{\pi}{\log \rho}$. Choose $\rho > 1$ so that $m(A) = \frac{\pi}{\log \rho}$ and identify A with $\mathbb{C}_+/\Lambda_\rho$. Lift f to a Λ_ρ -equivariant mapping from \mathbb{C}_+ to $C_n(\mathbb{C})/\mathcal{S}_n$ which we denote by f . For each $\theta \in (0, \pi)$ the mapping $[1, \rho] \ni r \rightarrow f(re^{i\theta}) \in C_n(\mathbb{C})/\mathcal{S}_n$ defines a closed curve that represents \hat{b} , in particular, we have the equality $f(e^{i\theta}) = f(\rho e^{i\theta})$ for the initial and terminating point. Take the closed curve obtained for $\theta = 0$. Consider it as a loop in $C_n(\mathbb{C})/\mathcal{S}_n$ with base point $E_n = f(i) = f(i\rho)$. Choose a point $(z_1, \dots, z_n) \in C_n(\mathbb{C})$ for which $\mathcal{P}_{\text{sym}}((z_1, \dots, z_n)) = E_n$. Choose a smooth self-diffeomorphism ψ of \mathbb{P}^1 which is the identity outside a large disc containing E_n^0 and maps ∞ to ∞ , and $\frac{j-1}{n}$ to z_j for $j = 1, \dots, n$. In particular, $\psi : \mathbb{C} \setminus E_n^0 \rightarrow \mathbb{C} \setminus E_n$. Denote by $\tilde{f} : \mathbb{C}_+ \rightarrow C_n(\mathbb{C})$ the lift of f to $C_n(\mathbb{C})$ for which $\tilde{f}(i\rho) = (z_1, \dots, z_n)$, i.e. $\mathcal{P}_{\text{sym}} \circ \tilde{f} = f$, $\tilde{f}(i\rho) = (z_1, \dots, z_n)$. The mapping $\mathcal{P}_A \circ \tilde{f}$ is a holomorphic mapping from \mathbb{C}_+ to $C_n(\mathbb{C})/\mathcal{A}$. (For the definition of \mathcal{P}_A see chapter 2.) Now we lift $\mathcal{P}_A \circ \tilde{f}$ with respect to the holomorphic projection $\mathcal{P}_T : \mathcal{T}(0, n+1) \rightarrow C_n(\mathbb{C})/\mathcal{A}$. We take the lift \mathcal{F} for which $\mathcal{F}(i\rho) = [\psi] \in \mathcal{T}(0, n+1)$. \mathcal{F} is a holomorphic map, $\mathcal{F} : \mathbb{C}_+ \rightarrow \mathcal{T}(0, n+1)$, such that $\mathcal{P}_T \circ \mathcal{F} = \mathcal{P}_A \circ \tilde{f}$. We have the following commutative diagram:

$$\begin{array}{ccccc}
& & \mathcal{F} & \longrightarrow & \mathcal{T}(0, n+1) \\
& & \searrow & & \downarrow \mathcal{P}_{\mathcal{T}} \\
& & \tilde{f} & \longrightarrow & C_n(\mathbb{C}) \xrightarrow{\mathcal{P}_{\mathcal{A}}} C_n(\mathbb{C})/\mathcal{A} \\
& & \downarrow \mathcal{P}_{\text{sym}} & & \\
\mathbb{C}_+ & \xrightarrow{f} & C_n(\mathbb{C})/\mathcal{S}_n & & \\
\downarrow & & & & \\
A \cong \mathbb{C}_+/\Lambda_\rho & \longrightarrow & C_n(\mathbb{C})/\mathcal{S}_n & &
\end{array}$$

FIGURE 4.1

Consider the restriction $\mathcal{F} \mid [i, i\rho]$. We want to relate the values $\mathcal{F}(i)$ and $\mathcal{F}(i\rho)$ at the endpoints using the fact that $\mathcal{F} \mid [i, i\rho]$ is obtained from a geometric braid by lifting and projecting. This will allow us to apply Royden's theorem.

Start with the geometric braid $[1, \rho] \ni t \rightarrow f(it) \in C_n(\mathbb{C})/\mathcal{S}_n$. The braid is given by a real analytic, hence, smooth mapping. Recall that the self-diffeomorphism ψ^{-1} of \mathbb{C} acts diagonally on $C_n(\mathbb{C})$, the action descends to $C_n(\mathbb{C})/\mathcal{S}_n$ and is denoted again by ψ^{-1} . Consider the smooth geometric braid $[1, \rho] \ni t \rightarrow \psi^{-1}(f(it)) \in C_n(\mathbb{C})/\mathcal{S}_n$. Its base point is E_n^0 . We consider it as geometric braid in the cylinder $[1, \rho] \times R\mathbb{D}$ for a large positive number R . ($R\mathbb{D}$ denotes the disc of radius R and center zero in \mathbb{C} .) The geometric braid $t \rightarrow \psi^{-1}(f(it))$ represents a braid b in \mathcal{B}_n in the same conjugacy class \hat{b} as the geometric braid $t \rightarrow f(it)$, $t \in [1, \rho]$. Let φ^t , $t \in [1, \rho]$, be a smooth parametrizing isotopy for $\psi^{-1}(f(it))$, $t \in [1, \rho]$. So for each $t \in [1, \rho]$ we have $\varphi^t \in \text{Hom}^+(R\mathbb{D}; \partial R\mathbb{D})$, $\varphi^\rho = \text{id}$ and

$$(4.1) \quad \text{ev}_{E_n^0} \varphi^t = \psi^{-1}(f(it)), \quad t \in [1, \rho].$$

The diffeomorphism φ^1 represents a mapping class $\mathbf{m}_b \in \mathfrak{M}(R\mathbb{D}; \partial(R\mathbb{D}), E_n^0)$ corresponding to the braid b . Denote for each $t \in [1, \rho]$ by φ_∞^t the extension of φ^t to \mathbb{P}^1 by the identity outside $R\mathbb{D}$. Each φ_∞^t is a diffeomorphism of \mathbb{P}^1 . Write $\varphi^1 = \varphi_b$ and $\varphi_\infty^1 = \varphi_{b, \infty}$. Let $\varphi_{b, \infty}^*$ be the modular transformation induced on $\mathcal{T}(0, n+1)$ by $\varphi_{b, \infty}$.

LEMMA 4.1. $\mathcal{F}(i) = \varphi_{b, \infty}^*(\mathcal{F}(i\rho))$.

Proof. By (4.1) we have $\text{ev}_{E_n^0} \psi \circ \varphi_\infty^t = f(it)$, $t \in [1, \rho]$. Both, $\tilde{f}(it)$ and $e_n(\psi \circ \varphi_\infty^t)$, $t \in [1, \rho]$, lift $f(it)$, $t \in [1, \rho]$. Both mappings lift $E_n = f(i\rho)$ to (z_1, \dots, z_n) . For $e_n(\psi \circ \varphi_\infty^t)$ this follows from the definition of ψ and the fact that $\varphi_\infty^\rho = \text{id}$. Hence, $e_n(\psi \circ \varphi_\infty^t) = \tilde{f}(it)$, $t \in [1, \rho]$. Using also Lemma 2.1 we obtain the following commutative diagram:

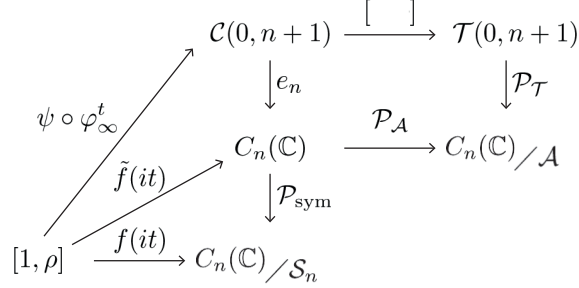


FIGURE 4.2

The diagrams Figure 4.1 and Figure 4.2 show that both, $\mathcal{F}(it)$ and $[\psi \circ \varphi_\infty^t]$, lift $\mathcal{P}_\mathcal{A} \circ \tilde{f}^t$ to a mapping from $[1, \rho]$ to the Teichmüller space $\mathcal{T}(0, n+1)$. Moreover, by the choice of the lift \mathcal{F} we have $\mathcal{F}(i\rho) = [\psi]$. Since $\varphi_\infty^\rho = \text{id}$, we may write $\mathcal{F}(i\rho) = [\psi \circ \varphi_\infty^\rho]$, which is the value of $[\psi \circ \varphi_\infty^t]$ for $t = \rho$. Therefore, the two lifts coincide:

$$(4.2) \quad \mathcal{F}(it) = [\psi \circ \varphi_\infty^t], \quad t \in [1, \rho].$$

In particular,

$$(4.3) \quad \mathcal{F}(i) = [\psi \circ \varphi_\infty^1] = [\psi \circ \varphi_{b,\infty}] = \varphi_{b,\infty}^*([\psi]) = \varphi_{b,\infty}^*(\mathcal{F}(i\rho)).$$

The lemma is proved. \square

The key ingredient for obtaining the upper bound for the conformal module is Royden's theorem on equality of the Kobayashi and the Teichmüller metric on the Teichmüller space $\mathcal{T}(0, n+1)$. Let d_{hyp} be the hyperbolic metric $\frac{|dz|}{2y}$ on \mathbb{C}_+ . By Royden's Theorem 2.3 the mapping \mathcal{F} is a contraction from $(\mathbb{C}_+, d_{\text{hyp}})$ to $(\mathcal{T}(0, n+1), d_\mathcal{T})$. In particular,

$$(4.4) \quad d_\mathcal{T}(\mathcal{F}(\rho i), \mathcal{F}(i)) \leq d_{\text{hyp}}(\rho i, i) = \frac{1}{2} \log \rho.$$

For the left hand side the following inequality holds

$$(4.5) \quad \begin{aligned} d_\mathcal{T}(\mathcal{F}(\rho i), \mathcal{F}(i)) &= d_\mathcal{T}(\mathcal{F}(\rho i), \varphi_{b,\infty}^*(\mathcal{F}(\rho i))) \\ &\geq \inf_{\tau \in \mathcal{T}(0, n+1)} d_\mathcal{T}(\tau, \varphi_{b,\infty}^*(\tau)) = L(\varphi_{b,\infty}^*). \end{aligned}$$

We obtained the following. If a conjugacy class \hat{b} of n -braids is represented by a holomorphic mapping $f : A \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$ of an annulus A of conformal module $m(A) = \frac{\pi}{\log \rho}$ into the symmetrized configuration space $C_n(\mathbb{C})/\mathcal{S}_n$ then

$$\frac{\pi}{2} \cdot \frac{1}{m(A)} = \frac{1}{2} \log \rho \geq L(\varphi_{b,\infty}^*)$$

We proved the following proposition.

PROPOSITION 4.1. *Let $\hat{b} \in \hat{\mathcal{B}}_n$ be a conjugacy class of n -braids. Let \mathfrak{m}_b be the mapping class group of a large n -punctured disc corresponding to a representative $b \in \hat{b}$ and let $\mathfrak{m}_{b,\infty} = \mathcal{H}_\infty(\mathfrak{m}_b)$ be the respective mapping class in $\mathfrak{M}(\mathbb{P}^1; \infty, E_n)$ of the n -punctured Riemann sphere. Consider for a representative $\varphi_{b,\infty}$ of $\mathfrak{m}_{b,\infty}$, its*

modular transformation $\varphi_{b,\infty}^*$ on $\mathcal{T}(0, n+1)$, and its translation length $L(\varphi_{b,\infty}^*)$. Then

$$(4.6) \quad L(\varphi_{b,\infty}^*) \leq \frac{1}{2} \log \rho = \frac{\pi}{2} \frac{1}{\mathcal{M}(\hat{b})}.$$

Assume now that the conjugacy class \hat{b} is irreducible, hence, $\varphi_{b,\infty}$ is irreducible and therefore $\varphi_{b,\infty}^*$ is either elliptic or hyperbolic. By Corollary 2.2 there is an absolutely extremal self-homeomorphism $\tilde{\varphi}_{b,\infty}$ of the Riemann surface $\mathbb{C} \setminus E_n^1$ which is obtained from $\varphi_{b,\infty}$ by isotopy and conjugation. More precisely, there is a homeomorphism $w : \mathbb{C} \setminus E_n^0 \rightarrow \mathbb{C} \setminus E_n^1$ and a self-homeomorphism $\hat{\varphi}_{b,\infty}$ of $\mathbb{C} \setminus E_n^0$ which is isotopic to $\varphi_{b,\infty}$ on $\mathbb{C} \setminus E_n^0$ so that $\tilde{\varphi}_{b,\infty} = w \circ \hat{\varphi}_{b,\infty} \circ w^{-1} : \mathbb{C} \setminus E_n^1 \rightarrow \mathbb{C} \setminus E_n^1$ is absolutely extremal. $\tilde{\varphi}_{b,\infty}$ is pseudo-Anosov if $\varphi_{b,\infty}^*$ is hyperbolic and conformal if $\varphi_{b,\infty}^*$ is elliptic. For $\tilde{\varphi}_{b,\infty}$ we have by Theorem 3.2 and Corollary 2.2

$$(4.7) \quad \frac{1}{2} \log K(\tilde{\varphi}_{b,\infty}) = L(\varphi_{b,\infty}^*) = h(\tilde{\varphi}_{b,\infty}) = h(\widehat{\mathfrak{m}_{b,\infty}}).$$

Theorem 4 provides an isotopy of $\tilde{\varphi}_{b,\infty}$ through self-homeomorphisms of $\mathbb{C} \setminus E_n^1$ to a homeomorphism $\tilde{\varphi}_b$ which is the identity outside the disc $R\overline{\mathbb{D}}$ and has the same entropy as $\tilde{\varphi}_{b,\infty}$:

$$(4.8) \quad h(\tilde{\varphi}_b) = h(\tilde{\varphi}_{b,\infty}).$$

Then $w^{-1} \circ \tilde{\varphi}_b \circ w$ is isotopic to $\varphi_{b,\infty}$ through self-homeomorphisms of $\mathbb{C} \setminus E_n^0$ and is the identity outside the disc $R\overline{\mathbb{D}}$. By Theorem 2.1 the mapping class \mathfrak{m}_b associated to b , and the mapping class of $w^{-1} \circ \tilde{\varphi}_b \circ w \mid R\overline{\mathbb{D}}$ in $\mathfrak{M}(R\overline{\mathbb{D}}; \partial(R\overline{\mathbb{D}}), E_n^0)$ differ by a power of a Dehn twist about a circle in $R\mathbb{D}$ of large radius. By Corollary 3.2 the entropies of the two mapping classes are equal:

$$(4.9) \quad \begin{aligned} h(\hat{b}) = h(b) &= \inf\{h(\varphi) : \varphi \in \mathfrak{m}_b\} \\ &= \inf\{h(\varphi) : \varphi \in \mathfrak{m}_{w^{-1} \circ \tilde{\varphi}_b \circ w \mid R\overline{\mathbb{D}}}\} \leq h(w^{-1} \circ \tilde{\varphi}_b \circ w \mid R\overline{\mathbb{D}}). \end{aligned}$$

Hence we obtain from (4.7) and (4.9)

$$h(\hat{b}) \leq h(w^{-1} \circ \tilde{\varphi}_b \circ w) = h(\tilde{\varphi}_b) = h(\tilde{\varphi}_{b,\infty}) = L(\varphi_{b,\infty}^*).$$

By (4.6) we have $h(\hat{b}) \leq \frac{\pi}{2} \frac{1}{\mathcal{M}(\hat{b})}$.

We proved the following proposition.

PROPOSITION 4.2. *For each irreducible conjugacy class of braids $\hat{b} \in \hat{\mathcal{B}}_n$ the inequality*

$$h(\hat{b}) \leq \frac{\pi}{2} \frac{1}{\mathcal{M}(\hat{b})}$$

holds.

Proof of Theorem 1. The lower bound for the conformal module. The irreducible case.

Take a conjugacy class of a n -braids $\hat{b} \in \hat{\mathcal{B}}_n$. Let $b \in \mathcal{B}_n$ be a braid with base point E_n^0 representing \hat{b} . Let φ_b be a self-homeomorphism of $\overline{\mathbb{D}} \setminus E_n^0$ which represents the mapping class $\mathbf{m}_b = \Theta_n(b) \in \mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$ of b . Let $\varphi_{b,\infty} \in \text{Hom}^+(\mathbb{C}; \emptyset, E_n^0)$ be the extension of φ_b to the whole plane which is the identity outside the unit disc.

Assume that the class \hat{b} is irreducible, i.e. $\varphi_{b,\infty}$ is irreducible. By Theorem 2.8 the induced modular transformation $\varphi_{b,\infty}^*$ on the Teichmüller space $\mathcal{T}(0, n+1)$ is either elliptic or hyperbolic.

Consider first the elliptic case. By Theorem 2.7 this is equivalent to the fact that there is a periodic conformal map $\psi_{b,\infty} \in \text{Hom}^+(\mathbb{C}; \emptyset, E_n)$ which is obtained from $\varphi_{b,\infty}$ by isotopy and conjugation. Here $E_n \subset \mathbb{C}$ is a set consisting of n points. The mapping $\psi_{b,\infty}$ is a Möbius transformation that fixes ∞ and, being periodic, has a fixed point on \mathbb{P}^1 different from ∞ . Conjugate this fixed point to zero. We obtain that $\psi_{b,\infty}$ is conjugate by a conformal mapping to multiplication by a root of unity $\omega = e^{2\pi i \frac{p}{q}}$. We may assume that the mapping $\psi_{b,\infty}$ itself has the form $\psi_{b,\infty}(z) = e^{2\pi i \frac{p}{q}} z$, for integer numbers p and q , $q \neq 0$. Since $\psi_{b,\infty}(E_n) = E_n$, the set E_n is either equal to $\left\{r, r e^{\frac{2\pi i}{n}}, \dots, r e^{\frac{2\pi i(n-1)}{n}}\right\}$ for some $r > 0$, and in this case $\omega = e^{\frac{2\pi i p}{n}}$ or to $E_n = \left\{0, r, r e^{\frac{2\pi i}{n-1}}, \dots, r e^{\frac{2\pi i(n-2)}{n-1}}\right\}$ for some $r > 0$, and in this case $\omega = e^{\frac{2\pi i p}{n-1}}$.

Consider the universal covering

$$\mathbb{C} \ni \zeta = \xi + i\eta \rightarrow e^{\xi+i\eta} \in \mathbb{C}^*$$

of the annulus $\mathbb{C}^* = \mathbb{C} \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < \infty\}$.

Denote by $\tilde{\mathcal{F}}$ the following holomorphic mapping from \mathbb{C} into the space \mathcal{A} of complex affine mappings,

$$(5.1) \quad \tilde{\mathcal{F}}(\zeta) = \mathbf{a}(\zeta) \in \mathcal{A}, \text{ where } \mathbf{a}(\zeta)(z) = e^{\frac{p}{q}(\xi+i\eta)} \cdot z.$$

Notice that

$$(5.2) \quad \mathbf{a}(0)(z) = z \quad \text{and} \quad \mathbf{a}(2\pi i)(z) = e^{2\pi i \frac{p}{q}} z = \psi_{b,\infty}(z).$$

The evaluation map

$$(5.3) \quad \mathbb{C} \ni \zeta \rightarrow \text{ev}_{E_n} \mathbf{a}(\zeta) \in C_n(\mathbb{C}) / \mathcal{S}_n$$

is holomorphic and is periodic with period $2\pi i$. Hence, the evaluation map induces a holomorphic map \mathcal{F} from \mathbb{C}^* to $C_n(\mathbb{C}) / \mathcal{S}_n$. \mathcal{F} represents a conjugacy class $\hat{b}_1 \in \hat{\mathcal{B}}_n$. By Theorem 2.1 the mapping classes corresponding to the braids $b_1 \in \hat{b}_1$ and $b \in \hat{b}$

differ by a power of a Dehn twist. Replacing \mathfrak{p} by $\mathfrak{p} + 2\pi k\mathfrak{q}$ for a suitable integral number k we may achieve that $\mathcal{F} : \mathbb{C}^* \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$ represents the conjugacy class $\hat{b} \in \hat{\mathcal{B}}_n$.

We proved the following proposition for the elliptic case.

PROPOSITION 5.1. *Let $\hat{b} \in \hat{\mathcal{B}}_n$ be a conjugacy class of braids. Suppose that \hat{b} is irreducible. Denote by r the number $r = \frac{\pi}{2} \frac{1}{h(\hat{b})}$ and denote by A_r the annulus $A_r = \left\{ z \in \mathbb{C} : \frac{1}{\sqrt{r}} < |z| < \sqrt{r} \right\}$. Then \hat{b} can be represented by a holomorphic map $f : A_r \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$.*

In the elliptic case $r = \infty$, hence $\sqrt{r} = \infty$, $\frac{1}{\sqrt{r}} = 0$ and $A_r = \mathbb{C}^*$. Notice that Proposition 5.1 is stronger than Theorem 1 in the irreducible case. Indeed, the Proposition 5.1 asserts that in the irreducible case the supremum in Definition 1 is attained. Moreover, if the supremum is infinite it is attained on the punctured complex plane rather than on the punctured disc.

Proof of Proposition 5.1 in the hyperbolic case. Take a braid $b \in \hat{b}$ with base point $E_n^0 = \left\{ 0, \frac{1}{n}, \dots, \frac{n-1}{n} \right\}$. Let φ_b be a representative of the mapping class $\mathfrak{m}_b \in \mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$ and $\varphi_{b,\infty} \in \text{Hom}^+(\mathbb{C}; \emptyset, E_n^0)$ the extension of φ_b to \mathbb{C} which is the identity outside the unit disc \mathbb{D} . Assume that the modular transformation $\varphi_{b,\infty}^*$ is hyperbolic. Denote $X_0 = \mathbb{C} \setminus E_n^0$. It will be convenient to work with $\varphi_{b,\infty}^{-1}$ rather than with $\varphi_{b,\infty}$. By Corollary 2.2 and Corollary 3.1 there exists a complex structure $w_0 : X_0 \rightarrow X \stackrel{\text{def}}{=} w_0(X_0)$ and an absolutely extremal map $\psi_{b,\infty}^{-1} : X \rightarrow X$ such that $\psi_{b,\infty}^{-1} = w_0 \circ \varphi^{-1} \circ w_0^{-1}$ for a self-homeomorphism φ^{-1} of X_0 which is isotopic to $\varphi_{b,\infty}^{-1}$ on X_0 . Moreover, by Corollary 3.2 $h(\psi_{b,\infty}^{-1}) = h(\widehat{\mathfrak{m}_{\varphi_{b,\infty}^{-1}}}) = h(\widehat{b^{-1}})$. Note that $h(\hat{b}) = h(\widehat{b^{-1}})$. We identify again self-homeomorphisms of punctured surfaces with the respective self-homeomorphisms of closed surfaces with distinguished points.

The Beltrami differential of the self-homeomorphism $\psi_{b,\infty}^{-1}$ of X has the form $-k \frac{\bar{\phi}}{|\phi|}$ for a meromorphic quadratic differential ϕ on \mathbb{P}^1 which is holomorphic on $\mathbb{C} \setminus E_n$ and has at worst simple poles at the points of $E_n \cup \{\infty\}$. Here

$$E_n = w_0(E_n^0) = \left\{ w_0(0), w_0\left(\frac{1}{n}\right), \dots, w_0\left(\frac{n-1}{n}\right) \right\},$$

and $k = \frac{K-1}{K+1}$, where K is the quasiconformal dilatation of $\psi_{b,\infty}$ (and hence of $\psi_{b,\infty}^{-1}$). Recall that by Theorem 3.2 we have $\frac{1}{2} \log K = h(\psi_{b,\infty}^{-1})$.

Let $\mathcal{D}_{\phi,X}$ be the Teichmüller disc in $\mathcal{T}(X)$ which is associated to the quadratic differential ϕ on X (see chapter 2).

The modular transformation $(\psi_{b,\infty}^{-1})^* = (\psi_{b,\infty}^*)^{-1}$ leaves $\mathcal{D}_{\phi,X}$ invariant. Indeed, $\psi_{b,\infty}$ has Beltrami differential $k \frac{\bar{\phi}}{|\phi|}$. For $\mu_z = z \frac{\bar{\phi}}{|\phi|}$, $z \in \mathbb{D}$, we let W^{μ_z} be the normalized solution on X of the Beltrami equation for μ_z (see Definition 2.1). By Lemma 9.1 of [26] the mapping $W^{\mu_z} \circ \psi_{b,\infty}^{-1}$ is a Teichmüller mapping with Beltrami differential $\text{const} \cdot \phi$ and quasiconformal dilatation $\left| \frac{z-k}{1-zk} \right|$. Hence $[W^{\mu_z} \circ \psi_{b,\infty}^{-1}] = (\psi_{b,\infty}^*)^{-1}([W^{\mu_z}])$ has the form $\{\mu_{z'}\}$ for some $z' \in \mathbb{D}$. If z is real then also z' is real.

Notice that the conformal structure w_0 realizes a canonical isomorphism between the Teichmüller space $\mathcal{T}(X)$ and the canonical Teichmüller space $\mathcal{T}(X_0) = \mathcal{T}(0, n+1)$. Indeed, associate to each conformal structure w on X the conformal structure $w \circ w_0$ on X_0 . Its class $[w \circ w_0]$ in $\mathcal{T}(0, n+1)$ depends only on the class $[w]$ in $\mathcal{T}(X)$. Put $w_0^*([w]) = [w \circ w_0]$, $[w] \in \mathcal{T}(X)$. The mapping $w_0^* : \mathcal{T}(X) \rightarrow \mathcal{T}(0, n+1)$ gives the canonical isomorphism. w_0^* is a holomorphic mapping.

Denote the image $w_0^*(\mathcal{D}_{\phi, X})$ by \mathcal{D}_{ϕ}^0 . \mathcal{D}_{ϕ}^0 is a holomorphic disc in $\mathcal{T}(0, n+1)$. Let \mathfrak{e}_{ϕ} be the mapping

$$(5.4) \quad \mathbb{D} \ni z \xrightarrow{\mathfrak{e}_{\phi}} w_0^*(\{\mu^z\}) \in \mathcal{D}_{\phi}^0 \subset \mathcal{T}(0, n+1).$$

The mapping $(\varphi_{b, \infty}^*)^{-1}$ leaves \mathcal{D}_{ϕ}^0 invariant.

Consider the mapping

$$(5.5) \quad \mathcal{E}_{\phi} : \mathbb{C}_+ \rightarrow \mathcal{D}_{\phi}^0, \quad \mathcal{E}_{\phi} = \mathfrak{e}_{\phi} \circ \mathfrak{c}^{-1},$$

where $\mathfrak{c} : \mathbb{D} \rightarrow \mathbb{C}_+$ is the conformal mapping $\mathfrak{c}(z) = i \frac{1+z}{1-z}$. The mapping

$$(5.6) \quad \Lambda = \mathcal{E}_{\phi}^{-1} \circ (\varphi_{b, \infty}^*)^{-1} \circ \mathcal{E}_{\phi} : \mathbb{C}_+ \rightarrow \mathbb{C}_+$$

is a holomorphic automorphism of \mathbb{C}_+ . Since $\mathfrak{c}_{\phi}^{-1} \circ (\varphi_{b, \infty}^*)^{-1} \circ \mathfrak{e}_{\phi}$ fixes the real axis and maps 0 to $-k$, the mapping Λ fixes the imaginary axis and maps i to $i \frac{1+k}{1-k} = iK$. Hence

$$(5.7) \quad \Lambda(\zeta) = \Lambda_K(\zeta) = K\zeta, \quad \zeta \in \mathbb{C}_+.$$

The annulus $A = \mathbb{C}_+ / \Lambda$ has conformal module $\frac{\pi}{2 \log K^{\frac{1}{2}}}$. Using the holomorphic mapping $\mathcal{E}_{\phi} : \mathbb{C}_+ \rightarrow \mathcal{T}(0, n+1)$ we want to find a holomorphic mapping $f : \mathbb{C}_+ / \Lambda \rightarrow C_n(\mathbb{C}) / \mathcal{A}$ which represents \hat{b} .

Let $\mathcal{U}(\varepsilon) = \{\zeta \in \mathbb{C}_+ : 1 - \varepsilon < |\zeta| < (1 + \varepsilon)K\}$ be a neighbourhood of the set $\{\zeta \in \mathbb{C}_+ : 1 \leq |\zeta| \leq K\}$ in \mathbb{C}_+ . Recall that $\mathcal{T}(0, n+1)$ is the universal covering of

$$C_n(\mathbb{C}) / \mathcal{A} \cong \{0\} \times \left\{ \frac{1}{n} \right\} \times C_{n-2}(\mathbb{C} \setminus \{0, \frac{1}{n}\}),$$

and the projection map $\mathcal{P}_{\mathcal{T}} : \mathcal{T}(0, n+1) \rightarrow C_n(\mathbb{C}) / \mathcal{A}$ is holomorphic. Here \mathcal{A} denotes the set of complex affine mappings. Consider the map $\mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi} : \mathcal{U}(\varepsilon) \rightarrow C_n(\mathbb{C}) / \mathcal{A}$. For $\zeta \in \mathbb{C}_+$ we identify the point $\mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}(\zeta)$ with a point

$$(5.8) \quad \mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}(\zeta) \cong \left(0, \frac{1}{n}, z_3(\zeta), \dots, z_n(\zeta)\right) \in \{0\} \times \left\{ \frac{1}{n} \right\} \times C_{n-2}(\mathbb{C} \setminus \{0, \frac{1}{n}\}).$$

For each holomorphic map $\mathfrak{A} : \mathcal{U}(\varepsilon) \rightarrow \mathcal{A}$ we denote by $f_{\mathfrak{A}} : \mathcal{U}(\varepsilon) \rightarrow C_n(\mathbb{C})$ the mapping

$$(5.9) \quad \begin{aligned} \mathcal{U}(\varepsilon) \ni \zeta &\xrightarrow{f_{\mathfrak{A}}} \left(\mathfrak{A}(\zeta)(0), \mathfrak{A}(\zeta)\left(\frac{1}{n}\right), \mathfrak{A}(\zeta)(z_3(\zeta)), \dots, \mathfrak{A}(\zeta)(z_n(\zeta)) \right) \\ &= \mathfrak{A}(\zeta)(\mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}(\zeta)) \in C_n(\mathbb{C}), \end{aligned}$$

which assigns to each point $\zeta \in \mathcal{U}(\varepsilon)$ the result of the diagonal action of the map $\mathfrak{A}(\zeta)$ on the point $\mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}(\zeta) \cong (0, \frac{1}{n}, z_3(\zeta), \dots, z_n(\zeta))$. For each holomorphic

mapping $\mathfrak{A} : \mathcal{U}(\varepsilon) \rightarrow \mathcal{A}$ the mapping $f_{\mathfrak{A}} : \mathcal{U}(\varepsilon) \rightarrow C_n(\mathbb{C})$ is a holomorphic section over $\mathcal{U}(\varepsilon)$ of the map $\mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}$ with respect to the projection $\mathcal{P}_{\mathcal{A}}$, i.e.

$$(5.10) \quad \mathcal{P}_{\mathcal{A}} f_{\mathfrak{A}} = \mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}.$$

Put $\mathcal{U}_1(\varepsilon) = \{\zeta \in \mathbb{C}_+ : 1 - \varepsilon < |\zeta| < 1 + \varepsilon\}$ for $\varepsilon > 0$. The following two lemmas imply Proposition 5.1.

LEMMA 5.1. *There exists a holomorphic map $\mathfrak{A} : \mathcal{U}(\varepsilon) \rightarrow \mathcal{A}$ such that the mapping $f(\zeta) \stackrel{\text{def}}{=} \mathcal{P}_{\text{sym}} f_{\mathfrak{A}}(\zeta)$, $\zeta \in \mathcal{U}(\varepsilon)$, has the following property:*

$$f(K\zeta) = f(\zeta) \quad \text{for } \zeta \in \mathcal{U}_1(\varepsilon).$$

Lemma 5.1 implies that f defines a holomorphic map from the annulus $A = \mathbb{C}_+ / \Lambda_K$ to $C_n(\mathbb{C}) / \mathcal{S}_n$.

Here is the diagram of the mappings

$$\begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ & \mathcal{U}(\varepsilon) & & C_n(\mathbb{C}) & \xrightarrow{\mathcal{P}_{\text{sym}}} & C_n(\mathbb{C}) / \mathcal{S}_n \\ & \searrow & f_{\mathfrak{A}} & \downarrow \mathcal{P}_{\mathcal{A}} & \\ \mathcal{U}(\varepsilon) & \xrightarrow{\mathcal{E}_{\phi}} & \mathcal{T}(0, n+1) & \xrightarrow{\mathcal{P}_{\mathcal{T}}} & C_n(\mathbb{C}) / \mathcal{A} \end{array}$$

FIGURE 5.1

LEMMA 5.2. *For the mapping f in Lemma 5.1 the family $f(it)$, $t \in [1, K]$, defines a closed path in $C_n(\mathbb{C}) / \mathcal{S}_n$ in the free isotopy class $\widehat{\Delta_n^{2\ell} b}$ for some $\ell \in \mathbb{Z}$.*

To prove the Proposition 5.1 we consider the mapping $\mathfrak{a}_{\ell}(\zeta)(f(\zeta))$, $\zeta \in \mathcal{U}(\varepsilon)$. Here f is the mapping of Lemma 5.1 and

$$(5.11) \quad \mathfrak{a}_{\ell}(\zeta)(z) = e^{-\frac{2\pi i \ell}{\log K} \cdot \log \frac{\zeta}{i}} \cdot z, \quad \zeta \in \mathcal{U}(\varepsilon), \quad z \in \mathbb{C}.$$

The mapping defines a holomorphic mapping from $A = \mathbb{C}_+ / \Lambda_K$ to $C_n(\mathbb{C}) / \mathcal{S}_n$. When ζ ranges over $[i, iK]$ the function $\log \frac{\zeta}{i}$ ranges over $[0, \log K]$ and $\mathfrak{a}_{\ell}(\zeta)$, $\zeta \in [i, iK]$, gives $-\ell$ full twists of \mathbb{C} . Hence, by Lemma 5.2 the path $\mathfrak{a}_{\ell}(\zeta)(f(\zeta))$, $\zeta \in [i, iK]$, represents the free isotopy class \hat{b} . The conformal module $m(A) = m(\mathbb{C}_+ / \Lambda_K)$ equals

$$\frac{\pi}{2} \frac{1}{\log \frac{K}{2}} = \frac{\pi}{2} \frac{1}{h(\widehat{b^{-1}})} = \frac{\pi}{2} \frac{1}{h(\hat{b})}.$$

Proposition 5.1 is proved. \square

Proof of Lemma 5.1. First we want to express $\mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}(K\zeta)$ by $\mathcal{P}_{\mathcal{T}} \circ \mathcal{E}_{\phi}(\zeta)$, $\zeta \in \mathbb{C}_+$. For this purpose we represent the holomorphic family $\mathcal{E}_{\phi}(\zeta) \in \mathcal{T}(0, n+1)$, $\zeta \in \mathbb{C}_+$, by a smooth family $W_{\phi}(\zeta)$, $\zeta \in \mathbb{C}_+^n$, of quasiconformal homeomorphisms from X_0 onto other Riemann surfaces. For each $\zeta \in \mathbb{C}_+$ let $z = \mathfrak{c}^{-1}(\zeta) \in \mathbb{D}$, and let W^{μ_z} be the normalized solution on X of the Beltrami equation for μ_z . Then with w_0 as above, $W^{\mu_z} \circ w_0$, $z \in \mathbb{D}$, is a smooth family of quasiconformal homeomorphisms

from X_0 onto other Riemann surfaces. Precompose these homeomorphisms with a smooth family of Möbius transformations so that the compositions fix $0, \frac{1}{n}$ and ∞ pointwise. Letting $\zeta = \mathfrak{c}(z), z \in \mathbb{D}$, we obtain a smooth family $W_\phi(\zeta), \zeta \in \mathbb{C}_+$, of quasiconformal homeomorphisms of \mathbb{P}^1 which fix the points $0, \frac{1}{n}$ and ∞ and map the set of distinguished points E_n^0 onto another set of distinguished points of \mathbb{P}^1 . (As above we identify homeomorphisms of punctured surfaces with homeomorphisms of closed surfaces with distinguished points.) For each $\zeta \in \mathbb{C}_+$ the mapping $W_\phi(\zeta)$ represents $\mathcal{E}_\phi(\zeta)$. Using the identification (5.8) we have

$$(5.12) \quad e_n(W_\phi(\zeta)) = \mathcal{P}_\mathcal{T} \circ \mathcal{E}_\phi(\zeta), \quad \zeta \in \mathbb{C}_+.$$

By (5.6) and (5.7)

$$(5.13) \quad \mathcal{E}_\phi(K\zeta) = (\varphi_{b,\infty}^*)^{-1}(\mathcal{E}_\phi(\zeta)), \quad \zeta \in \mathbb{C}_+.$$

Hence, the mapping $W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1} \in QC(X_0)$ is a representative of $\mathcal{E}_\phi(K\zeta)$. The map $\varphi_{b,\infty}$ permutes the points of the set E_n^0 as follows. Label the points in E_n^0 as $z_j = \frac{j-1}{n}$. Put $s_b = \tau(b)$. The permutation s_b acts on the set of the first n natural numbers by the formula $s_b(1, 2, \dots, n) = (s_b(1), \dots, s_b(n))$. Denote by S_b the induced action on the set of labeled points: $S_b(z_j) = z_{s_b(j)}$. Then $\varphi_{b,\infty}(z_j) = S_b(z_j), j = 1, \dots, n$. Hence,

$$(5.14) \quad e_n(W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1}) = S_b^{-1}(e_n(W_\phi(\zeta))), \quad \zeta \in \mathbb{C}_+.$$

Since by (5.13) for each $\zeta \in \mathbb{C}_+$ the two mappings $W_\phi(K\zeta)$ and $W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1}$ are Teichmüller equivalent, also the two mappings

$$\mathfrak{A}(K\zeta)(W_\phi(K\zeta)) \quad \text{and} \quad \mathfrak{A}(\zeta)(W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1})$$

are Teichmüller equivalent for any mapping $\mathfrak{A} : \mathbb{C}_+ \rightarrow \mathcal{A}$.

LEMMA 5.3. *There exists a holomorphic mapping $\mathfrak{A} : \mathcal{U}(\varepsilon) \rightarrow \mathcal{A}$ such that for each $\zeta \in \mathcal{U}_1(\varepsilon)$ the first two coordinates of the two n -tuples $e_n(\mathfrak{A}(K\zeta)(W_\phi(K\zeta)))$ and $e_n(\mathfrak{A}(\zeta)(W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1}))$ coincide.*

End of proof of Lemma 5.1. Take the mapping $\mathfrak{A} : \mathcal{U}(\varepsilon) \rightarrow \mathcal{A}$ of Lemma 5.3. By Lemma 2.2

$$(5.15) \quad e_n(\mathfrak{A}(K\zeta)(W_\phi(K\zeta))) = e_n(\mathfrak{A}(\zeta)(W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1})), \quad \zeta \in \mathcal{U}_1(\varepsilon).$$

By (5.9) and (5.12) and since the action of $\mathfrak{A}(\zeta)$ commutes with e_n we have

$$(5.16) \quad f_\mathfrak{A}(\zeta) = \mathfrak{A}(\zeta)(\mathcal{P}_\mathcal{T} \circ \mathcal{E}_\phi(\zeta)) = \mathfrak{A}(\zeta)(e_n(W_\phi(\zeta))) = e_n(\mathfrak{A}(\zeta)(W_\phi(\zeta))), \quad \zeta \in \mathcal{U}(\varepsilon).$$

By (5.14) and because the action of $\mathfrak{A}(\zeta)$ commutes with e_n and with $(S_b)^{-1}$ we have

$$e_n(\mathfrak{A}(\zeta)(W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1})) = (S_b)^{-1}(e_n(\mathfrak{A}(\zeta)(W_\phi(\zeta)))), \quad \zeta \in \mathcal{U}_1(\varepsilon).$$

We obtain from (5.15) and (5.16) that $f_\mathfrak{A}(K\zeta) = (S_b)^{-1}(f_\mathfrak{A}(\zeta))$, hence,

$$(5.17) \quad \mathcal{P}_{\text{sym}} f_\mathfrak{A}(K\zeta) = \mathcal{P}_{\text{sym}} f_\mathfrak{A}(\zeta), \quad \zeta \in \mathcal{U}_1(\varepsilon).$$

Lemma 5.1 is proved. \square

Proof of Lemma 5.3. We have to find the mapping \mathfrak{A} so that for $\zeta \in \mathcal{U}_1(\varepsilon)$ the first two coordinates of

$$\mathfrak{A}(K\zeta)(e_n(W_\phi(K\zeta))) \quad \text{and} \quad \mathfrak{A}(\zeta)(e_n(W_\phi(\zeta) \circ \varphi_{b,\infty}^{-1})) = \mathfrak{A}(\zeta)(S_b^{-1}(e_n(W_\phi(\zeta))))$$

coincide. Write

$$e_n(W_\phi(\zeta)) = (z_1(\zeta), z_2(\zeta), z_3(\zeta), \dots, z_n(\zeta))$$

with $z_1(\zeta) \equiv 0$, $z_2(\zeta) \equiv \frac{1}{n}$. We denote $s = s_b$ for short. The condition reads then as follows:

$$(5.18) \quad \begin{aligned} \mathfrak{A}(K\zeta)(0) &= \mathfrak{A}(\zeta)(z_{s(1)}(\zeta)), \\ \mathfrak{A}(K\zeta)\left(\frac{1}{n}\right) &= \mathfrak{A}(\zeta)(z_{s(2)}(\zeta)), \quad \zeta \in \mathcal{U}_1(\varepsilon). \end{aligned}$$

For $\zeta \in \mathbb{C}_+$ we write $\mathfrak{A}(\zeta)$ in the form $\mathfrak{A}(\zeta)(z) = \frac{z-b(\zeta)}{a(\zeta)}$, $z \in \mathbb{C}$, for holomorphic functions a and b in \mathbb{C}_+ with $a \neq 0$ in \mathbb{C}_+ . For any number $\varepsilon > 0$ and any function q on $\mathcal{U}(\varepsilon)$ we use the notation $q^K(\zeta)$, $\zeta \in \mathcal{U}_1(\varepsilon)$, for the function $q^K(\zeta) = q(K\zeta)$. The equations (5.18) can be written as

$$(5.19) \quad \begin{aligned} -\frac{b^K}{a^K} &= \frac{z_{s(1)} - b}{a}, \\ \frac{\frac{1}{n} - b^K}{a^K} &= \frac{z_{s(2)} - b}{a}, \quad \zeta \in \mathcal{U}_1(\varepsilon). \end{aligned}$$

Put $\chi \stackrel{\text{def}}{=} n(z_{s(2)} - z_{s(1)})$. χ is an analytic function on \mathbb{C}_+ , $\chi \neq 0$ on \mathbb{C}_+ . The equations (5.19) can be rewritten as

$$(5.20) \quad \begin{aligned} \frac{a}{a^K} &= \chi, \\ \frac{b^K}{a^K} - \frac{b}{a} &= -\frac{z_{s(1)}}{a}, \quad \zeta \in \mathcal{U}_1(\varepsilon). \end{aligned}$$

It is enough to solve the following problem (*) for each sufficiently small number $\varepsilon > 0$:

(*) *For a holomorphic function β on $\mathcal{U}_1(4\varepsilon)$ find a holomorphic function α on $\mathcal{U}(\varepsilon)$ such that $\alpha^K - \alpha = \beta$ on $\mathcal{U}_1(\varepsilon)$.*

Indeed, for the first equation let β_1 be the restriction to $\mathcal{U}(16\varepsilon)$ of a holomorphic branch of $\log \chi$ on \mathbb{C}_+ . Let α_1 be a holomorphic function on $\mathcal{U}(4\varepsilon)$ which is a solution of the problem (*) on $\mathcal{U}_1(4\varepsilon)$ with this choice of β_1 . Then $a = e^{\alpha_1}$ solves the first equation on $\mathcal{U}_1(4\varepsilon)$. Put $\beta_2 = -\frac{z_{s(1)}}{a}$ on $\mathcal{U}_1(4\varepsilon)$. Let α_2 be a holomorphic function on $\mathcal{U}(\varepsilon)$ which is a solution on $\mathcal{U}_1(\varepsilon)$ of the problem (*) with β_2 . Take $\frac{b}{a} = \alpha_2$. The mapping $\mathfrak{A}(\zeta)(z) = \frac{z-b(\zeta)}{a(\zeta)}$, $z \in \mathbb{C}$, $\zeta \in \mathcal{U}(\varepsilon)$, is a solution of the problem of Lemma 5.3.

Problem (*) can be solved since for the annulus $A = \mathbb{C}_+/\Lambda_K$ the second cohomology $H^2(A, \mathbb{Z})$ with coefficients in the set \mathbb{Z} of integral numbers is trivial (see [13], [14], Proposition V.1.8).

This is essentially an elementary $\bar{\partial}$ -problem. For convenience of the reader we provide the short solution.

Let ϱ be a smooth function on $(1 - \varepsilon, K(1 + \varepsilon))$ which equals one in a neighbourhood of $(-1 - \varepsilon, 1 + \varepsilon]$ and equals zero in a neighbourhood of $[1 + 3\varepsilon, (1 + \varepsilon)K)$. Consider the function v ,

$$v(\zeta) = \begin{cases} -\varrho(|\zeta|)\beta(\zeta), & |\zeta| \in (1 - \varepsilon, 1 + 4\varepsilon), \quad \zeta \in \mathbb{C}_+, \\ 0, & |\zeta| \in [1 + 4\varepsilon, K(1 + \varepsilon)), \quad \zeta \in \mathbb{C}_+. \end{cases}$$

The function v is of class C^∞ in \mathcal{U} since $\varrho(|\zeta|) = 0$ on $\{1 + 3\varepsilon < |\zeta| < 1 + 4\varepsilon\}$. Moreover, for $\zeta \in \mathcal{U}_1(\varepsilon)$ we have $v(K\zeta) - v(\zeta) = \beta(\zeta)$.

The $(0, 1)$ -form $\omega = \bar{\partial}v$ on \mathcal{U} vanishes on $\{\zeta \in \mathbb{C}_+ : |\zeta| \in (1 - \varepsilon, 1 + \varepsilon)\} \cup \{\zeta \in \mathbb{C}_+ : |\zeta| \in (K(1 - \varepsilon), K(1 + \varepsilon))\}$ since v is holomorphic there. Hence, we can consider ω as a smooth form on the annulus $A = \mathbb{C}_+ / \Lambda_K$ (for which $\{1 < |\zeta| \leq K\} \cap \mathbb{C}_+$ is a fundamental polygon). Let u be a solution of the equation $\bar{\partial}u = \omega$ on the annulus A . We may consider u as a Λ_K -equivariant function on \mathbb{C}_+ . The function $\alpha \stackrel{\text{def}}{=} v - u$ on $\mathcal{U}(\varepsilon)$ solves the problem $(*)$. \square

Proof of Lemma 5.2. For the mapping f we have

$$\begin{aligned} f(\zeta) = \mathcal{P}_{\text{sym}} f\mathfrak{A}(\zeta) &= \mathcal{P}_{\text{sym}} e_n(\mathfrak{A}(\zeta)(W_\phi(\zeta))) \\ (5.21) \quad &= \text{ev}_{E_n^0}(\mathfrak{A}(\zeta)(W_\phi(\zeta))), \quad \zeta \in \mathcal{U}(\varepsilon). \end{aligned}$$

Consider the continuous family of homeomorphisms

$$(5.22) \quad \varphi(\zeta) = \mathfrak{A}(\zeta)(W_\phi(\zeta)), \quad \zeta \in \mathcal{U}(\varepsilon).$$

Composing each $\varphi(\zeta)$, $\zeta \in \mathcal{U}(\varepsilon)$, with the inverse of $\mathfrak{A}(i)$, we may assume that $\varphi(i) = W_\phi(i)$. Recall that $\mathcal{E}_\phi(i) = \mathfrak{e}_\phi(0)$ corresponds to $\{\mu^0\} = [\text{id}]$ in $\mathcal{T}(X)$. Using the isomorphism

$$\begin{aligned} QC(X) \ni w &\rightarrow w \circ w_0 \in QC(X_0), \\ \mathcal{T}(X) \ni [w] &\rightarrow w_0^*([w]) \in \mathcal{T}(X_0) = \mathcal{T}(0, n+1), \end{aligned}$$

we obtain that $W_\phi(i) = w_0 \in QC(X_0)$.

The homeomorphism $\varphi(iK) = \mathfrak{A}(iK)(W_\phi(iK))$ is an element of the Teichmüller class

$$[W_\phi(iK)] = [W_\phi(i) \circ \varphi_{b,\infty}^{-1}] = [w_0 \circ \varphi_{b,\infty}^{-1}] \in \mathcal{T}(X_0), \quad X_0 = \mathbb{C} \setminus E_n^0.$$

By Lemma 5.3 and Lemma 2.2 $\varphi(iK)$ is isotopic to $w_0 \circ \varphi_{b,\infty}^{-1}$ on X_0 . Put $\tilde{\varphi}(it) = w_0^{-1} \circ \varphi(it) \circ \varphi_{b,\infty}$, $t \in [1, K]$. We obtain a geometric braid

$$\text{ev}_{E_n^0} \tilde{\varphi}(it), \quad t \in [1, K],$$

which is free isotopic to $f(t)$, $t \in [1, K]$.

Note that $\tilde{\varphi}(i) = \varphi_{b,\infty}$, and $\tilde{\varphi}(iK)$ is isotopic to id through self-homeomorphisms of X_0 . Using the isotopy of $\tilde{\varphi}(iK)$ to the identity we obtain a continuous family $\tilde{\varphi}_1(it)$, $t \in [1, K]$, so that the geometric braids $\text{ev}_{E_n^0} \tilde{\varphi}_1(it)$, $t \in [1, K]$, and $\text{ev}_{E_n^0} \tilde{\varphi}(it)$, $t \in [1, K]$, are free isotopic, $\tilde{\varphi}_1(i) = \tilde{\varphi}(i) = \varphi_{b,\infty}$, and $\tilde{\varphi}_1(iK) = \text{id}$. Let $R\mathbb{D}$ be a large disc which contains $\text{ev}_{E_n^0} \tilde{\varphi}_1(it)$, for $t \in [1, K]$.

Consider the family $\tilde{\varphi}_1(it)$, $t \in [1, K]$, as a homeomorphism of $[1, K] \times \mathbb{C}$ which preserves all fibers $\{t\} \times \mathbb{C}$. Make an isotopy of this homeomorphism fixing it on $[1, K] \times R\mathbb{D}$ so that we obtain a fiber preserving homeomorphism which is the identity outside $[1, K] \times R_1\mathbb{D}$ for a number $R_1 > R$. (We may use smoothing and a vector field argument.) This gives a new family $\varphi^\#(it)$, $t \in [1, K]$. The construction may be done so that $\varphi^\#(iK) = \text{id}$. Hence, $\varphi^\#(it) \mid R_1\mathbb{D}$, $t \in [1, K]$, is a parametrizing isotopy for a geometric braid in the same conjugacy class as $f(it)$, $t \in [1, K]$. Since $\varphi^\#(i)$ is isotopic to $\tilde{\varphi}(i)$ on $\mathbb{C} \setminus E_n^0$ and $\tilde{\varphi}(i) = \varphi_{b,\infty}$, the mappings $\varphi^\#(i) \mid R_1\mathbb{D}$ and $\varphi_{b,\infty} \mid R_1\mathbb{D}$ in $\text{Hom}(R_1\mathbb{D}; \partial(R_1\mathbb{D}), E_n^0)$ differ by a power of a Dehn twist. Hence, the class of $f(it)$, $t \in [1, K]$, differs from b by a power of Δ_n^2 . \square

CHAPTER 6

Reducible braids and irreducible components

This chapter prepares the proof of Theorem 1 in the case of reducible conjugacy classes of braids. On one hand the proof of the theorem uses a decomposition of reducible elements of $\widehat{\mathfrak{M}}(\mathbb{P}^1; \infty, E_n)$ into irreducible nodal components (see chapter 2). On the other hand the proof uses a decomposition of reducible braids into irreducible components. We will establish here the relation between the irreducible components of braids and the irreducible components of mapping classes, and relate their invariants. The decomposition of reducible objects into irreducible components can be considered as “analysis”. To obtain the invariants of the reducible objects, we need “synthesis”. For braids this means the following. Knowing the conjugacy classes of the irreducible components of a braid, give an explicit expression of a geometric braid that represents the conjugacy class of the reducible braid.

Reducible mapping classes of the punctured disc and a partial order of cycles. Let $b \in \mathcal{B}_n$ be a braid, let $\mathfrak{m}_b \in \mathfrak{M}(\overline{\mathbb{D}}; \partial\mathbb{D}, E_n)$ be the corresponding mapping class. Suppose \mathfrak{m}_b is reducible. Let φ_b be a homeomorphism representing \mathfrak{m}_b and let $\mathcal{C} = \{C_1, \dots, C_k\}$ be an admissible system of curves in $\mathbb{D} \setminus E_n$ which completely reduces φ_b . In particular, φ_b leaves the union $\bigcup_{j=1}^k C_j$ invariant, and also

leaves the complement $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ invariant.

The Jordan curve theorem induces a partial order on the set of connected components of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$, and also induces a partial order on the cycles of φ_b . The partial order is described as follows. It will be needed for the “analysis” and “synthesis” of reducible braids.

Denote by S^1 the connected component of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ which has $\partial\mathbb{D}$ as a boundary component. Call S^1 the component of first generation. Denote by $S^{2,j}$, $j = 1, \dots, k'_2$, the connected components of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ which have a boundary component in common with S^1 . By the Jordan curve theorem each $S^{2,j}$ has exactly one such boundary component. We call it the exterior boundary component of $S^{2,j}$ and denote it by $\partial_{\mathfrak{E}} S^{2,j}$, $j = 1, \dots, k'_2$. The set $\partial_{\mathfrak{I}} S^{2,j} = \partial S^{2,j} \setminus \partial_{\mathfrak{E}} S^{2,j}$ is called the interior boundary of $S^{2,j}$. The components $S^{2,j}$ are called the components of second generation. The mapping φ_b permutes the components of second generation along cycles. Indeed, φ_b fixes the component S^1 (setwise) since it fixes its boundary component $\partial\mathbb{D}$. Hence φ_b fixes $\partial_{\mathfrak{I}} S^1$ setwise and therefore the union $\bigcup_{j=1}^{k'_2} S^{2,j}$

is invariant under φ_b . Denote the φ_b -cycles of components of second generation by $\text{cyc}^{2,i}$, $i = 1, \dots, k_2$. Here $k_2 \leq k'_2$.

The connected components of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ of generation ℓ and the cycles of generation ℓ are defined by induction as follows. Consider the union of all connected components of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ of generation not exceeding $\ell - 1$. Take its closure $\overline{\bigcup_{1 \leq \ell' \leq \ell-1} S^{\ell',j}}$, which is the closure of a domain $Q_{\ell-1} \subset \mathbb{D}$. $Q_{\ell-1}$ is the union of all components of generation not exceeding $\ell - 1$ and the exterior boundaries of all components of generation between 2 and $\ell - 1$. The connected components of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ which share a boundary component with $Q_{\ell-1}$ other than $\partial\mathbb{D}$ are called the components of generation ℓ and are labeled by $S^{\ell,j}$, $j = 1, \dots, k'_\ell$. Each $S^{\ell,j}$ has exactly one boundary component in common with $Q_{\ell-1}$. We call this part of the boundary of $S^{\ell,j}$ the exterior boundary of $S^{\ell,j}$ and denote it by $\partial_{\mathfrak{E}} S^{\ell,j}$. The remaining set $\partial S^{\ell,j} \setminus \partial_{\mathfrak{E}} S^{\ell,j}$ is denoted by $\partial_{\mathfrak{I}} S^{\ell,j}$ and is called the interior boundary of $S^{\ell,j}$. By induction we see that the mapping φ_b fixes $Q_{\ell-1}$ setwise. Hence, φ leaves the union $\bigcup_{j=1}^{k'_\ell} S^{\ell,j}$ invariant, and therefore permutes the $S^{\ell,j}$ along cycles. The φ_b -cycles of components of generation ℓ are denoted by $\text{cyc}^{\ell,i}$, $i = 1, \dots, k_i$. Here $k_i \leq k'_i$.

The process terminates after we obtained sets of generation N for some finite number N . Each of the curves in $\mathcal{C} \cup \{\partial\mathbb{D}\}$ is the exterior boundary of a single connected component of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$. The exterior boundary $\partial_{\mathfrak{E}} S^{\ell,j}$ of each component of generation ℓ is surrounded by exactly $\ell - 1$ of the curves in $\mathcal{C} \cup \{\partial\mathbb{D}\}$. Each of the surrounding curves is the exterior boundary of a component $S^{\ell',j'}$ of generation ℓ' , $\ell' = 1, \dots, \ell - 1$.

Associate to the class \mathfrak{m}_b of b the mapping class $\mathfrak{m}_{b,\infty} = \mathcal{H}_\infty(\mathfrak{m}_b)$ in $\mathfrak{M}(\mathbb{P}^1; \infty, E_n)$ (see (2.4), (2.5)). It will be convenient in the sequel to identify $\mathfrak{M}(\mathbb{P}^1; \infty, E_n)$ with a subset of $\mathfrak{M}(\mathbb{C} \setminus E_n)$. Identify the system of curves \mathcal{C} in \mathbb{D} with the identical system of curves in \mathbb{C} . Also, identify each component $S^{\ell,j}$, $\ell \geq 2$, with the identical set of \mathbb{C} . Instead of the component S^1 we consider the subset $S^1 \cup (\mathbb{C} \setminus \mathbb{D})$ of \mathbb{C} , which we denote again by S^1 if no confusion arises.

Let

$$w : \mathbb{C} \setminus E_n \rightarrow Y$$

be a continuous surjection onto a nodal surface Y associated to the isotopy class of the curve system \mathcal{C} in \mathbb{C} (see chapter 2). Denote by $\mathring{\mathfrak{m}}_b$ the isotopy class of self-mappings of the nodal surface Y determined by $\mathfrak{m}_{b,\infty}$ and w . Let $\widehat{\mathring{\mathfrak{m}}_b}$ be the conjugacy class of $\mathring{\mathfrak{m}}_b$.

The mapping w maps each component $S^{\ell,j} \setminus E_n$ of $\mathbb{C} \setminus (E_n \cup \bigcup C_j)$ homeomorphically onto a part $Y^{\ell,j}$ of Y . (Recall that a part of a nodal surface with set of nodes \mathcal{N} is a connected component of $Y \setminus \mathcal{N}$.) The correspondence between the components $S^{\ell,j} \setminus E_n$ and the parts $Y^{\ell,j}$ is a bijection. The mappings representing $\mathring{\mathfrak{m}}_b$ permute the parts of Y along cycles which we call nodal cycles and denote by $\text{cyc}^{\ell,i}$. The nodal cycles $\text{cyc}^{\ell,i}$ are in correspondence with the cycles $\text{cyc}^{\ell,i}$. We call

the conjugacy classes of the restrictions of $\overset{\circ}{\mathbf{m}}_b$ to the cycles $\overset{\circ}{\text{cyc}}^{\ell,i}$, i.e. the classes $\widehat{\overset{\circ}{\mathbf{m}}_b | \overset{\circ}{\text{cyc}}^{\ell,i}}$, the irreducible nodal components of the class $\widehat{\mathbf{m}_{b,\infty}}$. Notice, that the irreducible nodal components determine the class $\widehat{\mathbf{m}_{b,\infty}}$ only up to products of powers of some Dehn twists. But they are the only objects related to a decomposition of mappings which depend only on the isotopy class of the curve system \mathcal{C} and on the conjugacy class $\widehat{\mathbf{m}_{b,\infty}}$.

The irreducible components of braids. Pure braids. We will now describe the decomposition of reducible braids into irreducible components. To make the explanation more transparent we will first consider the case of pure braids.

Let $b \in \mathcal{B}_n$ be a pure braid, and let φ_b be a representative of the mapping class $\mathbf{m}_b \in \mathfrak{M}(\mathbb{D}; \partial\mathbb{D}, E_n)$ which is completely reduced by the admissible system \mathcal{C} of curves.

The homeomorphism φ_b fixes the set E_n pointwise. Hence φ_b fixes each curve $C_j \in \mathcal{C}$ setwise and fixes each connected component of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ setwise. Indeed,

each component $S^{N,j}$ of $\mathbb{D} \setminus \bigcup_{j=1}^k C_j$ of the last generation contains points of E_n by the definition of an admissible set of curves. Hence, φ_b fixes setwise each such component, and also fixes setwise its boundary $\partial S^{N,j} = \partial_{\mathfrak{E}} S^{N,j}$. By induction, suppose φ_b fixes setwise all components of generation at least $\ell > 1$, and fixes setwise each boundary component of each of it. Each component $S^{\ell-1,j}$ contains either points of E_n or has non-empty interior boundary. All components of $\partial_j S^{\ell-1,j}$ are also boundary components of an $S^{\ell,j'}$ and, each such component is fixed setwise by the induction hypothesis. Hence, φ fixes each $S^{\ell-1,j}$ setwise and fixes setwise each boundary component of $S^{\ell-1,j}$.

Replacing φ_b by an isotopic mapping we may assume that φ_b fixes each curve C_j pointwise.

Let $\varphi_t \in \text{Hom}(\overline{\mathbb{D}}; \partial\mathbb{D})$, $t \in [0, 1]$ be a parametrizing isotopy for a geometric braid representing b , i.e. $\varphi_0 = \varphi_b$, $\varphi_1 = \text{id}$ and the geometric braid $\varphi_t(E_n)$, $t \in [0, 1]$, represents b . We identify this geometric braid with a mapping $g : [0, 1] \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$.

The irreducible component $b(1)$ of a pure braid b associated to S^1 . Let $E^1 = E_n \cap S^1$ be the set of distinguished points contained in the component S^1 . Let $C^{2,j}$, $j = 1, \dots, k_2$, be the set of interior boundary components of S^1 , labeled so that $C^{2,j}$ is the exterior boundary of the component $S^{2,j}$, $j = 1, \dots, k_2$. Denote by $\delta^{2,j} \subset \mathbb{D}$ the topological disc bounded by $C^{2,j}$. For each j we pick a point $z^{2,j} \in \delta^{2,j} \cap E_n$. Such a point exists by the definition of an admissible system of curves and an induction argument. Since the braid is pure we have $\varphi(z^{2,j}) = z^{2,j}$. Put $\mathcal{E}^1 = \{z^{2,1}, \dots, z^{2,k_2}\}$. Associate to S^1 the geometric braid

$$(6.1) \quad \varphi_t(E^1 \cup \mathcal{E}^1), \quad t \in [0, 1].$$

Its isotopy class is a pure braid which we denote by $b(1)$. The braid $b(1)$ carries the following additional information. The set of its strands is divided into two subsets, E_1 and \mathcal{E}_1 . There is a bijective correspondence of the second subset \mathcal{E}_1 onto the set of connected components of $\mathbb{C} \setminus \bigcup_{\mathcal{C}} C$ of second generation.

Intuitively, the braid $b(1)$ is obtained from the braid b by discarding all strands with initial point not in the set $E^1 \cup \mathcal{E}^1$.

We can also look at the braid $b(1)$ in the following way. Consider the “partially thickened” geometric braid

$$(6.2) \quad \varphi_t \left(E^1 \cup \bigcup_{j=1}^{k_2} \overline{\delta^{2,j}} \right), \quad t \in [0, 1],$$

and take the deformation retraction of each of the cylinders $\varphi_t \left(\overline{\delta^{2,j}} \right)$ to a strand of b contained in this cylinder.

Let $n(1)$ be the number of points of $E^1 \cup \mathcal{E}^1$. Having in mind a label of the points in $E^1 \cup \mathcal{E}^1$ we identify the set $E^1 \cup \mathcal{E}^1$ with a subset of $C_{n(1)}(\mathbb{C})$ and identify the geometric braid (6.1) with a mapping from the unit interval into $C_{n(1)}(\mathbb{C})$. We write this mapping as

$$(6.3) \quad g^1 : [0, 1] \rightarrow C_{n(1)}(\mathbb{C}), \quad g^1(0) = g^1(1).$$

Here we label the coordinates of $C_{n(1)}(\mathbb{C})$ by the label of the initial points of the strands of the geometric braid (6.1). See Figure 6.1 for the geometric braid g^1 and the respective “partially thickened” geometric braid. In the figure the set E^1 consists of a single point. There are two components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ of generation 2, $S^{2,1}$ and $S^{2,2}$. The respective distinguished points are $z^{2,1}$ and $z^{2,2}$.

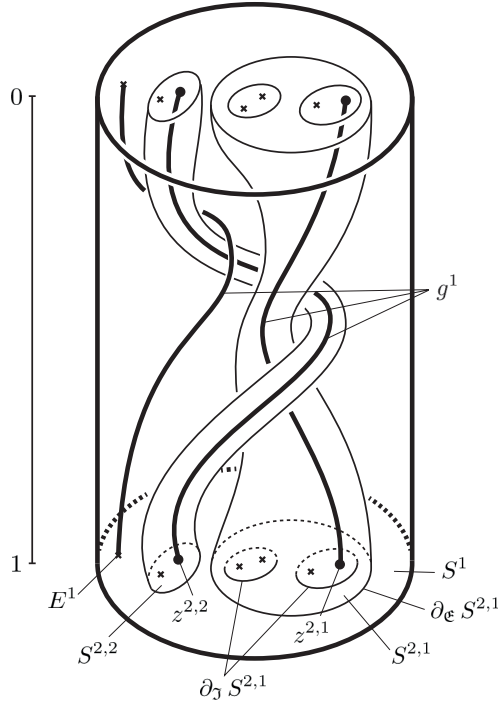


FIGURE 6.1

It will be useful to associate to the geometric braid g^1 the closed geometric braid

$$(6.4) \quad G^1(e^{2\pi it}) = g^1(t), \quad t \in [0, 1].$$

We obtain a mapping

$$G^1 : \partial \mathbb{D} = \{|z| = 1\} \rightarrow C_{n(1)}(\mathbb{C}).$$

The irreducible component $b(\ell, j)$ associated to $S^{\ell, j}$. Take a component $S^{\ell, j}$ of generation ℓ . Put $E^{\ell, j} = E \cap S^{\ell, j}$. Consider the interior boundary components of $S^{\ell, j}$. Each such boundary component is the exterior boundary of a component $S^{\ell+1, j'}$ of $\mathbb{P}^1 \setminus \bigcup_{i=1}^k C_i$ of generation $\ell + 1$. Let $\delta^{\ell+1, j'} \subset \mathbb{D}$ be the topological disc bounded by $\partial_{\mathbb{C}} S^{\ell+1, j'}$, and let $z^{\ell+1, j'}$ be a point in $E \cap \delta^{\ell+1, j'}$. Let $\mathcal{E}^{\ell, j}$ be the collection of $z^{\ell+1, j'}$ obtained in this way (one such point is assigned to each interior boundary component of $S^{\ell, j}$). Associate to $S^{\ell, j}$ the pure geometric braid

$$(6.5) \quad \varphi_t(E^{\ell, j} \cup \mathcal{E}^{\ell, j}), \quad t \in [0, 1].$$

The isotopy class of this geometric braid is the irreducible component $b(\ell, j)$ associated to $S^{\ell, j}$.

Intuitively, the braid $b(\ell, j)$ is obtained from the braid b by discarding all strands with initial point not in $E^{\ell, j} \cup \mathcal{E}^{\ell, j}$.

In still another way, consider the “partially thickened” geometric braid

$$(6.6) \quad \varphi_t \left(E^{\ell, j} \cup \bigcup_{j=1}^{k_\ell} \overline{\delta^{\ell, j}} \right), \quad t \in [0, 1],$$

and take the deformation retraction of each of the cylinders $\varphi_t(\overline{\delta^{\ell, j}})$ to a strand of b contained in this cylinder.

The braid $b(\ell, j)$ is equipped with the following additional information. The set of strands is divided into two subsets $E^{\ell, j}$ and $\mathcal{E}^{\ell, j}$. There is a bijection between the second set $\mathcal{E}^{\ell, j}$ and the subset of those connected components of generation $\ell + 1$ of $\mathbb{C} \setminus \bigcup_{i=1}^k C_i$ which are adjacent to $S^{\ell, j}$ (that is, of those connected components which share a boundary component with $S^{\ell, j}$).

Let $n(\ell, j)$ be the number of points of $E^{\ell, j} \cup \mathcal{E}^{\ell, j}$. We will identify the geometric braid (6.5) with a mapping $g^{\ell, j} : [0, 1] \rightarrow C_{n(\ell, j)}(\mathbb{C})$ from the unit interval into $C_{n(\ell, j)}(\mathbb{C})$, where the coordinates of $C_{n(\ell, j)}(\mathbb{C})$ are labeled by the label of the initial points of the strands of $b(\ell, j)$. Associate to $g^{\ell, j}$ the closed geometric braid

$$(6.7) \quad G^{\ell, j} : \partial \mathbb{D} \rightarrow C_{n_{\ell, j}}(\mathbb{C}), \quad G^{\ell, j}(e^{2\pi it}) = g^{\ell, j}(t).$$

We described the “analysis” of reducible pure braids, i.e. their decomposition into irreducible components.

We describe now the “synthesis”. In other words, suppose we know the conjugacy classes $\widehat{b(1)}$ and $\widehat{b(\ell, j)}$ for all ℓ and j , together with additional information

consisting in a bijection of a subset of the set of strands of $\widehat{b(\ell, j)}$ (respectively of $\widehat{b(1)}$) to the set of those connected components of $\mathbb{C} \setminus \cup_C C$ of generation $\ell + 1$ that are adjacent to $S^{\ell, j}$ (respectively to the set of connected components of generation 2). We want to recover the conjugacy class \hat{b} .

Start with the conjugacy class $\widehat{b(1)}$. Take a representing geometric braid for $b(1)$ and identify it with a mapping

$$f^1 : [0, 1] \rightarrow C_{n(1)}.$$

Denote the base point of the geometric braid by $E_f^1 \cup \mathcal{E}_f^1$ where the set \mathcal{E}_f^1 is in bijection with the set of connected components of $\mathbb{C} \setminus \cup_C C$ of second generation. Let $\mathcal{E}_f^1 = \{z_f^{2,1}, \dots, z_f^{2,k_2}\}$ where the point $z_f^{2,j}$ corresponds to the component $S^{2,j}$. Write $f^1 = (f_{2,1}, \dots, f_{2,k_2}, \dots)$, where $f_{2,j}$ is the component of the vector valued function f^1 with initial point $f_{2,j}(1) = z_f^{2,j}$ corresponding to $S^{2,1}$. We do not choose notation for the remaining components. They are labeled by their initial points which are contained in E_f^1 .

In the same way we represent each conjugacy class $\widehat{b(\ell, j)}$ by a geometric braid which we identify with a vector valued function $f^{\ell, j} : [0, 1] \rightarrow C_{n(\ell, j)}(\mathbb{C})$. The set of initial points of the geometric braid $f^{\ell, j}$ is $E_f^{\ell, j} \cup \mathcal{E}_f^{\ell, j}$. Here $\mathcal{E}_f^{\ell, j}$ corresponds to the set of components of $\mathbb{C} \setminus \cup_C C$ of generation $\ell + 1$ which share a boundary component with $S^{\ell, j}$ and the coordinates of $C_{n(\ell, j)}(\mathbb{C})$ are labeled by the label of the initial points of $f^{\ell, j}$.

We introduce an operation on the conjugacy classes of braids $\widehat{b(1)}$ and $\widehat{b(2, 1)}$ as follows.

Take a small positive number $\epsilon^{2,1}$ and consider the point $\epsilon^{2,1}(E_f^{2,1} \cup \mathcal{E}_f^{2,1}) \in C_{n(2,1)}(\mathbb{C})$. Denote by

$$(6.8) \quad z_f^{2,1} \boxplus \epsilon^{2,1}(E_f^{2,1} \cup \mathcal{E}_f^{2,1})$$

the point in $C_{n(2,1)}(\mathbb{C})$ obtained by adding the number $z_f^{2,1}$ to each component of $\epsilon^{2,1}(E_f^{2,1} \cup \mathcal{E}_f^{2,1})$.

Similarly, we denote by

$$(6.9) \quad f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}$$

the vector valued function on $[0, 1]$ obtained by adding for each $t \in [0, 1]$ the number $f_{2,1}(t)$ to the point $\epsilon^{2,1} f^{2,1}(t) \in C_{n(2,1)}(\mathbb{C})$. We identify (6.9) with a geometric braid. Notice that the initial point of this geometric braid is (6.8). If $\epsilon^{2,1}$ is small the geometric braid (6.9) is contained in a small neighbourhood of the strand of f^1 defined by $f_{2,1}$.

Replace the component $f_{2,1}$ of f^1 by the vector valued function $f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}$. Denote the resulting vector valued function by

$$(6.10) \quad f^1 \sqcup (f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}) : [0, 1] \rightarrow C_{n(1)+n(2,1)-1}(\mathbb{C}).$$

The positive number $\epsilon^{2,1}$ is chosen so small that the vector valued function (6.10) defines a pure geometric braid, in other words, so that the graphs of the components of (6.10) are pairwise disjoint.

The base point of this geometric braid is the point

$$(6.11) \quad z_f^1 \sqcup (z_f^{2,1} \boxplus \epsilon^{2,1}(E_f^{2,1} \cup \mathcal{E}_f^{2,1}))$$

which is obtained from z_f^1 by replacing its component $z_f^{2,1}$ by (6.8).

We want to relate the geometric braid (6.10) to the conjugacy class \hat{b} . Take the representing braid $b \in \hat{b}$ with base point $E_n \subset C_n(\mathbb{C})$, and choose the homeomorphism φ_b representing its mapping class as above. Let $\varphi_t \in \text{Hom}(\overline{\mathbb{D}}; \partial \mathbb{D})$, $t \in [0, 1]$, be a continuous family so that $\varphi_0 = \varphi_b$, $\varphi_1 = \text{id}$. The geometric braid

$$(6.12) \quad \varphi_t(E_n), \quad t \in [0, 1].$$

represents b .

We will define now a geometric braid denoted by $g^1 \sqcup g^{2,1}$ which is obtained from (6.12) by discarding a collection of strands, and will show that (6.9) is free isotopic to $g^1 \sqcup g^{2,1}$.

Let $E^1 \cup \mathcal{E}^1$, and $E^{2,1} \cup \mathcal{E}^{2,1}$ respectively, be the subsets of E_n corresponding to the components S^1 , and $S^{2,1}$ respectively.

Consider the geometric braid

$$(6.13) \quad \varphi_t(E^1 \cup (\mathcal{E}^1 \setminus \{z^{2,1}\}) \cup E^{2,1} \cup \mathcal{E}^{2,1}), \quad t \in [0, 1].$$

Identify

$$\varphi_t(E^1 \cup \mathcal{E}^1), \quad t \in [0, 1],$$

with a vector valued function

$$g^1 : [0, 1] \rightarrow \mathcal{C}_{n(1)}, \quad g^1 = (g_{2,1}, \dots, g_{2,k_2}),$$

and identify

$$\varphi_t(E^{2,1} \cup \mathcal{E}^{2,1})$$

with a vector valued function

$$g^{2,1} : [0, 1] \rightarrow \mathcal{C}_{n(2,1)}.$$

Then (6.13) can be identified with the vector valued function

$$(6.14) \quad g^1 \sqcup g^{2,1} : [0, 1] \rightarrow C_{n(1)+n(2,1)-1}(\mathbb{C}),$$

obtained by replacing the component $g_{2,1}$ of g^1 by the vector valued function $g^{2,1}$.

LEMMA 6.1. *The geometric braids (6.10) and (6.14) are free isotopic.*

Proof of Lemma 6.1 The geometric braids f^1 and g^1 represent braids in the same class $\widehat{b(1)}$ with base points $E_f^1 \cup \mathcal{E}_f^1$ and $E^1 \cup \mathcal{E}^1$ respectively. Hence the two geometric braids are free isotopic and by Proposition 2.4 there exists a continuous family $\varphi_t^s \in \text{Hom}^+(\overline{\mathbb{D}}; \partial \mathbb{D})$, $(t, s) \in [0, 1] \times [0, 1]$, with the following properties. For the family φ_t used in (6.12) we have $\varphi_t^0 = \varphi_t$. There is a continuous family $\psi^s \in \text{Hom}^+(\overline{\mathbb{D}}; \partial \mathbb{D})$, $s \in [0, 1]$, with $\psi^0 = \text{id}$, so that for each $s \in [0, 1]$ the evaluation

$$(6.15) \quad \varphi_t^s(\psi^s(E^1 \cup \mathcal{E}^1)), \quad t \in [0, 1]$$

defines a geometric braid $f^1(s)$ with base point $\psi^s(E^1 \cup \mathcal{E}^1)$ (see the notation of Remark 2.1). Moreover, the family $f^1(s)$ defined by (6.15) for $s \in [0, 1]$ is a free isotopy of geometric braids joining g^1 and f^1 ,

$$f^1(0) = g^1, \quad f^1(1) = f^1.$$

Notice that the family of evaluations on the whole set E_n , given for each $s \in [0, 1]$ by $\varphi_t^s(\psi^s(E_n))$, $t \in [0, 1]$, defines a free isotopy of geometric braids. It joins the

original geometric braid $\varphi_t(E_n) = \varphi_t^0(E_n)$, $t \in [0, 1]$, with base point E_n with the geometric braid

$$(6.16) \quad \varphi_t^1(\psi^1(E_n)), \quad t \in [0, 1],$$

with base point $\psi^1(E_n)$. Hence the geometric braid (6.16) also represents \hat{b} .

We replace φ_t by φ_t^1 and E_n by $\psi^1(E_n)$ and keep previous notation for the new objects. Also, we replace the sets $\delta^{\ell,j}$ by $\psi^1(\delta^{\ell,j})$, and replace the sets $E^1 \cup \mathcal{E}^1$ and $E^{\ell,j} \cup \mathcal{E}^{\ell,j}$ by the sets $\psi^1(E^1 \cup \mathcal{E}^1)$ and $\psi^1(E^{\ell,j} \cup \mathcal{E}^{\ell,j})$ and use previous notation. Let also g^1 and $g^{\ell,j}$ denote now the mappings obtained for the new objects.

After this replacement we have $f^1 = g^1$. By the definition of $g^{2,1}$ we have

$$(6.17) \quad g^{2,1}(t) \subset \varphi_t(\delta^{2,1}), \quad t \in [0, 1].$$

We may assume that $\epsilon^{2,1} > 0$ is so small that also

$$(6.18) \quad f^1(t) \sqcup (f_{2,1}(t) \boxplus \epsilon^{2,1} f^{2,1}(t)) \subset \varphi_t(\delta^{2,1}), \quad t \in [0, 1].$$

Notice that $\epsilon^{2,1} f^{2,1}$ is free isotopic to $f^{2,1}$, and $(f_{2,1} \boxplus \epsilon^{2,1} f^{2,1})$ is free isotopic to $\epsilon^{2,1} f^{2,1}$. Also, $g^{2,1}$ is free isotopic to $f^{2,1}$ (since they represent the same class $\widehat{b(2,1)}$). Hence, $f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}$ is free isotopic to $g^{2,1}$. Take a free isotopy of braids $g^s(t)$ joining the two geometric braids. Since both geometric braids, $g^{2,1}$ and $f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}$, are contained in

$$(6.19) \quad \bigcup_{t \in [0,1]} \{t\} \times \varphi_t(\delta^{2,1}),$$

the two geometric braids are free isotopic by an isotopy $g(s)$, $s \in [0, 1]$, which fixes the complement of the set (6.19) pointwise. Indeed, take $r \in (0, 1)$ close to 1 so that the free isotopy of braids $g(s)$, $s \in [0, 1]$, is contained in $[0, 1] \times r\mathbb{D}$ and $\delta^{2,1} \subset r\mathbb{D}$. Let $\delta^{2,1}_{\circ}$ be an open relatively compact topological disc in $\delta^{2,1}$ which contains $E^{2,1} \cup \mathcal{E}^{2,1}$ and $E_f^{2,1} \cup \mathcal{E}_f^{2,1}$. Let χ be a self-diffeomorphism of $[0, 1] \times \mathbb{D}$, which preserves each fiber $\{t\} \times \mathbb{D}$, is equal to the identity on $\partial\mathbb{D}$ and on $\bigcup_{t \in [0,1]} \{t\} \times \varphi_t(\delta^{2,1}_{\circ})$, and maps the set $[0, 1] \times (r\mathbb{D})$ onto $\bigcup_{t \in [0,1]} \{t\} \times \varphi_t(\delta^{2,1})$.

Then $\chi \circ g(0) = g(0) = g^{2,1}$ and $\chi \circ g(1) = g(1) = f^{2,1}$, and $\chi \circ g(s)$, $s \in [0, 1]$, is a free isotopy of braids contained in $\bigcup_{t \in [0,1]} \{t\} \times \varphi_t(\delta^{2,1})$ joining $g(0) = g^{2,1}$ with $g(1) = f^{2,1}$. Thus $g^1 \sqcup \chi \circ g(s)$, $s \in [0, 1]$, is a free isotopy of (6.10) and (6.14). The lemma is proved. \square

The free isotopy class of braids corresponding to (6.10), or equivalently to (6.14), is denoted by $\widehat{b(1)} \sqcup \widehat{b(2,1)}$.

Notice that we can look at the composed geometric braid $g^1 \sqcup g^{2,1}$ in the following intuitive way. Consider the “partially thickened” geometric braid

$$(6.20) \quad \varphi_t(E^1 \cup \overline{\delta^{2,1}}), \quad t \in [0, 1].$$

Insert the geometric braid $g^{2,1}$ into the cylinder $\left\{ \varphi_t(\overline{\delta^{2,1}}), t \in [0, 1] \right\}$, with trivialization defined by the inclusion $\left\{ \varphi_t(\overline{\delta^{2,1}}), t \in [0, 1] \right\} \subset [0, 1] \times \mathbb{D}$. Notice that a

free isotopy of $g^{2,1}$ inside the cylinder leads to a free isotopy of the composed geometric braid. Also, a free isotopy of the geometric braid g^1 together with a suitable deformation of the parametrizing isotopy leads to a free isotopy of the composed geometric braid.

We choose now by induction on $j = 1, \dots, k_2 - 1$, suitable representatives $f^{2,j+1}$ of $\widehat{b(2, j+1)}$ and compose

$$(6.21) \quad f^1 \sqcup (f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}) \sqcup \dots \sqcup (f_{2,j} \boxplus \epsilon^{2,j} f^{2,j})$$

with $f_{2,j+1} \boxplus \epsilon^{2,j+1} f^{2,j+1}$ for a suitable small positive number $\epsilon^{2,j+1}$.

The induction step of proof is to show that (6.21) is free isotopic to the evaluation of φ_t (see (6.12)) on a subset of E_n . By induction we may assume that there is an isotopy $\varphi_t \in \text{Hom}^+(\overline{\mathbb{D}}, \partial\mathbb{D})$, $t \in [0, 1]$, and a base point E_n such that the geometric braid $\varphi_t(E_n)$, $t \in [0, 1]$, represents the class \hat{b} which we want to recover, and with the respective choices of subsets of E_n , the geometric braid

$$\varphi_t(E^1 \cup (\mathcal{E}^1 \setminus \{z^{2,1}, \dots, z^{2,j}\}) \cup (E^{2,1} \cup \mathcal{E}^{2,1}) \cup \dots \cup (E^{2,j} \cup \mathcal{E}^{2,j})), \quad t \in [0, 1],$$

can be identified with the mapping (6.21). For $j = 1$ it may be achieved in the same way as it was done for f^1 . Namely, we apply again Proposition 2.4, now to the free isotopy classes of braids $g^1 \sqcup g^{2,1}$ and $f^1 \sqcup (f_{2,1} \boxplus \epsilon^{2,1} f^{2,1})$. We obtain a continuous family of elements of $\text{Hom}^+(\overline{\mathbb{D}}, \partial\mathbb{D})$ which we denote again by φ_t^s , $(t, s) \in [0, 1] \times [0, 1]$, and a family ψ^s as in Remark 2.1, such that for $s \in [0, 1]$ the braid $\varphi_t^s(\psi^s(E^1 \cup (\mathcal{E}^1 \setminus \{z^{2,1}\}) \cup E^{2,1} \cup \mathcal{E}^{2,1}))$, $t \in [0, 1]$, is free isotopic to $g^1 \sqcup g^{2,1}$, it is equal to it for $s = 0$ and is equal to $f^1 \sqcup (f_{2,1} \boxplus \epsilon^{2,1} f^{2,1})$ for $s = 1$. We replace again $\varphi_t = \varphi_t^0$ by φ_t^1 , replace E_n by $\psi^1(E_n)$, and replace $\delta^{2,1}$ by $\psi^1(\delta^{2,1})$. We denote the new objects by the previous letters.

Repeat the argument which allowed to compose f^1 with $f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}$. Now we compose (6.21) with $f_{2,j+1} \boxplus \epsilon^{2,j+1} f^{2,j+1}$ for a sufficiently small positive number $\epsilon^{2,j+1}$.

A finite number of analogous steps gives a geometric braid

$$(6.22) \quad f^1 \sqcup (f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}) \sqcup \dots \sqcup (f_{2,k_2} \boxplus \epsilon^{2,k_2} f^{2,k_2}).$$

which is free isotopic to

$$(6.23) \quad \varphi_t(E^1 \cup (\mathcal{E}^1 \setminus \{z^{2,1}, \dots, z^{2,j}\}) \cup (E^{2,1} \cup \mathcal{E}^{2,1}) \cup \dots \cup (E^{2,k_2} \cup \mathcal{E}^{2,k_2})), \quad t \in [0, 1].$$

We are looking now at the irreducible components $\widehat{b(\ell, j)}$ for $\ell > 2$. Recall that the conjugacy classes $\widehat{b(\ell, j)}$ are equipped with additional information which consists in a division of the set of strands into two subsets $E^{\ell,j}$ and $\mathcal{E}^{\ell,j}$ and a bijection of the second set $\mathcal{E}^{\ell,j}$ onto the set of those connected components of $\mathbb{C} \setminus \cup_C C$ of generation $\ell + 1$ which are adjacent to $S^{\ell,j}$. Note that for fixed number of the generation ℓ the union $\bigcup_{j=1}^{k_\ell} \mathcal{E}^{\ell,j}$ is in bijection to the set of all connected components of $\mathbb{C} \setminus \cup_C C$ of generation $\ell + 1$.

Let $\ell = 2$. Consider the union $\bigcup_{j=1}^{k_2} \mathcal{E}^{2,j}$ and label the points in the union by $(z^{3,1}, \dots, z^{3,k_3})$ according to the label of the connected components of $\mathbb{C} \setminus \cup_C C$ of generation 3. Denote the component of the mapping (6.22) corresponding to the point $z^{3,j}$ by $f_{3,j}$. Do a similar construction as we did for composing f^1 with geometric braids representing components of generation 2, to compose (6.22) with

the geometric braids $f_{3,j} \boxplus \epsilon^{3,j} f^{3,j}$ representing the components of generation 3. Here $f^{3,j}$ are suitable representatives of $\widehat{b(3,j)}$ and $\epsilon^{3,j}$ are small enough positive numbers, so that the respective composition is free isotopic to the geometric braid

$$\varphi_t((E^1)' \cup (E^2)' \cup (E^3)'), \quad t \in [0, 1],$$

where

$$\begin{aligned} (E^1)' &= (E^1 \cup \mathcal{E}^1) \setminus \{z^{2,1}, \dots, z^{2,j}\}, \\ (E^2)' &= ((E^{2,1} \cup \mathcal{E}^{2,1}) \cup \dots \cup (E^{2,k_2} \cup \mathcal{E}^{2,k_2})) \setminus \{z^{3,1}, \dots, z^{3,k_3}\}, \\ (E^3)' &= (E^{3,1} \cup \mathcal{E}^{3,1}) \cup \dots \cup (E^{3,k_3} \cup \mathcal{E}^{3,k_3}). \end{aligned}$$

Continue by induction until we reach $\ell = N$. We arrive at the geometric braid

$$(6.24) \quad f^1 \sqcup (f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}) \sqcup \dots \sqcup (f_{2,k_2} \boxplus \epsilon^{2,k_2} f^{2,k_2}) \sqcup \dots \sqcup (f_{N,k_N} \boxplus \epsilon^{N,k_N} f^{N,k_N}).$$

We denote the class represented by (6.24) by

$$(6.25) \quad \widehat{b(1)} \sqcup \widehat{b(2,1)} \sqcup \dots \sqcup \widehat{b(2,k_2)} \sqcup \dots \sqcup \widehat{b(N,k_N)}.$$

Notice that $\partial_{\mathcal{I}} S^{N,j} = \emptyset$ for each component $S^{N,j}$ of the last generation N . Hence, an easy induction shows that the conjugacy class of braids represented by (6.24) equals \hat{b} ,

$$(6.26) \quad \widehat{b(1)} \sqcup \widehat{b(2,1)} \sqcup \dots \sqcup \widehat{b(2,k_2)} \sqcup \dots \sqcup \widehat{b(N,k_N)} = \hat{b}.$$

We described the “synthesis” for reducible pure braids.

The following lemma describes the relation between the irreducible components of a pure braid and the irreducible nodal components of its mapping class.

As always, for a conjugacy class \hat{a} of braids and a representative $a \in \hat{a}$ we let $\mathfrak{m}_{a,\infty}$ be the image $\mathfrak{m}_a = \mathcal{H}(\mathfrak{m}_a)$ in the mapping class group of the n -punctured complex plane \mathbb{C} . Denote by $\widehat{\mathfrak{m}_{a,\infty}}$ its conjugacy class. The class $\widehat{\mathfrak{m}_{a,\infty}}$ depends only on \hat{a} .

LEMMA 6.2. *Let b be a pure braid and \mathfrak{m}_b its mapping class in $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n)$. Let \mathcal{C} be a system of curves in $\mathbb{D} \setminus E_n$ which completely reduces a representative φ_b of \mathfrak{m}_b . Let $S^{\ell,j}$, respectively S^1 , be the connected components of $\mathbb{C} \setminus \bigcup_{C \in \mathcal{C}} C$ and let Y be a nodal surface which is associated to the admissible system \mathcal{C} of curves in $\mathbb{C} \setminus E_n$. Let Y^1 , and $Y^{\ell,j}$ respectively, be the parts of Y corresponding to the components $S^1 \setminus E_n$ and $S^{\ell,j} \setminus E_n$, respectively.*

Then for each part Y^1 and $Y^{\ell,j}$ and for the associated irreducible components $b(1)$ and $b(\ell,j)$ of the braid we have the following equality

$$(6.27) \quad \widehat{\mathfrak{m}_{b(\ell,j),\infty}} = \widehat{\mathfrak{m}_{b,\infty}} \mid Y^{\ell,j}.$$

Respectively,

$$(6.28) \quad \widehat{\mathfrak{m}_{b(1),\infty}} = \widehat{\mathfrak{m}_{b,\infty}} \mid Y^1.$$

Note that the individual nodal surface Y depends on the individual curve system \mathcal{C} and on the continuous surjection $w : \mathbb{D} \setminus E_n \rightarrow Y$. Also, the mapping class $\mathring{\mathbf{m}}_{b,\infty}$ on Y induced by $\mathbf{m}_{b,\infty}$ and w depends on these objects. However, the conjugacy class $\widehat{\mathring{\mathbf{m}}_{b,\infty}}$ and also the conjugacy classes $\widehat{\mathring{\mathbf{m}}_{b,\infty} | Y^{\ell,j}}$ of the restrictions to the parts of the nodal surface depend only on the isotopy class of \mathcal{C} and on the class $\widehat{\mathbf{m}}_{b,\infty}$.

Proof of Lemma 6.2. The class on the right hand side of (6.27) is obtained as follows. Recall that the parts of a nodal surface Y with set of nodes \mathcal{N} are the connected components of $Y \setminus \mathcal{N}$. For each part $Y^{\ell,j}$ of the nodal surface Y and the respective connected component $S^{\ell,j} \setminus E_n$ of $\mathbb{C} \setminus (\bigcup_{C \in \mathcal{C}} C \cup E_n)$ the mapping $w | S^{\ell,j} \setminus E_n$ is a homeomorphism from $S^{\ell,j} \setminus E_n$ onto $Y^{\ell,j}$. Respectively, $w | S^1 \setminus E_n$ is a homeomorphism onto the part Y^1 . Notice that $Y^{\ell,j}$ (Y^1 , respectively) is a punctured sphere. For $\ell > 1$ the set of punctures is the union of the set the nodes $\mathcal{N} \cap \overline{Y^{\ell,j}}$ and the set $w(E^{\ell,j})$. The set of punctures of Y^1 is the union of the set of nodes $\mathcal{N} \cap \overline{Y^1}$, the point ∞ and the set $w(E^1)$.

Denote by $(S^{\ell,j})^{\text{Cl}}$ the finite point compactification of $S^{\ell,j}$ obtained by adding a point $z_{\infty,i}^{\ell,j}$ for each interior boundary component of $S^{\ell,j}$ and adding a point $z_{\infty,\infty}^{\ell,j}$ for the exterior boundary of $S^{\ell,j}$. Denote the (unordered) collection of these points by $Z_{\infty}^{\ell,j} = \{z_{1,\infty}^{\ell,j}, \dots, z_{k(\ell,j),\infty}^{\ell,j}, z_{\infty,\infty}^{\ell,j}\}$. We have the canonical homomorphism

$$(6.29) \quad \mathcal{H}_{\partial} : \mathfrak{M}(\overline{S^{\ell,j}}; E^{\ell,j} \cup \partial S^{\ell,j}) \rightarrow \mathfrak{M}((S^{\ell,j})^{\text{Cl}}; E^{\ell,j} \cup Z_{\infty}^{\ell,j})$$

which acts by restricting representing self-homeomorphisms of $\overline{S^{\ell,j}}$ to $S^{\ell,j}$ and extending the restrictions to self-homeomorphisms of $(S^{\ell,j})^{\text{Cl}}$.

For $\ell > 1$ the map $w | S^{\ell,j}$ extends to a homeomorphism $w^{\ell,j}$ of $(S^{\ell,j})^{\text{Cl}}$ onto the closure $\overline{Y^{\ell,j}}$ of $Y^{\ell,j}$ in \overline{Y} . The extended map maps the collection $Z_{\infty}^{\ell,j}$ onto the collection $\mathcal{N}^{\ell,j} = \mathcal{N} \cap \overline{Y^{\ell,j}}$ of nodes of Y contained in $\overline{Y^{\ell,j}}$. The mapping $w^{\ell,j}$ induces an isomorphism

$$(6.30) \quad (w^{\ell,j})^* : \mathfrak{M}((S^{\ell,j})^{\text{Cl}}; E^{\ell,j} \cup Z_{\infty}^{\ell,j}) \rightarrow \mathfrak{M}(\overline{Y^{\ell,j}}; w(E^{\ell,j}) \cup \mathcal{N}^{\ell,j})$$

by conjugation.

In the same way we put $Z_{\infty}^1 = \{z_{1,\infty}^1, \dots, z_{k(1),\infty}^1, z_{\infty,\infty}^1\}$, where $z_{\infty,\infty}^1$ is defined to be equal to ∞ . As for $\ell > 1$ we get an isomorphism

$$(6.31) \quad (w^1)^* : \mathfrak{M}((S^1)^{\text{Cl}}; E^1 \cup Z_{\infty}^1) \rightarrow \mathfrak{M}(\overline{Y^1}; w(E^1) \cup \mathcal{N}^1).$$

The mapping class groups on the right hand sides of (6.30) and (6.31) are identified with subsets of the mapping class groups $\mathfrak{M}(Y^{\ell,j})$, and $\mathfrak{M}(Y^1)$, respectively.

It will be convenient to express the mapping classes using the representative φ_b . We may assume that φ_b fixes pointwise $\partial \mathbb{D}$ and all curves in \mathcal{C} .

By the definition of the class $\mathring{\mathbf{m}}_{b,\infty}$ we have

$$(6.32) \quad \mathring{\mathbf{m}}_{b,\infty} | \overline{Y^{\ell,j}} = (w^{\ell,j})^* \circ \mathcal{H}_{\partial} \left(\mathbf{m} \left(\varphi_b | \overline{S^{\ell,j}} \right) \right).$$

Here $\mathbf{m} \left(\varphi_b | \overline{S^{\ell,j}} \right)$ denotes the mapping class of $\varphi_b | \overline{S^{\ell,j}}$ in $\mathfrak{M}(\overline{S^{\ell,j}}; E^{\ell,j} \cup \partial S^{\ell,j})$.

Respectively, we have

$$(6.33) \quad \mathring{\mathbf{m}}_{b,\infty} | \overline{Y^1} = (w^1)^* \circ \mathcal{H}_\partial \left(\mathbf{m} \left(\varphi_b | \overline{S^1} \right) \right) .$$

Consider the class on the right hand side of (6.32). By the construction of the braid $b(\ell, j)$ the class $\mathbf{m}_{b(\ell, j)}$ is the mapping class of the chosen representative φ_b of \mathbf{m}_b in $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D} \cup E^{\ell, j} \cup \mathcal{E}^{\ell, j})$. We can think of it also in the following way which makes the relation to the irreducible nodal component more transparent. Extend the restriction $\varphi_b | \overline{S^{\ell, j}}$ by the identity to all discs $\delta^{\ell+1, j'} \subset \mathbb{D}$ which are bounded by the interior boundary components of $S^{\ell, j}$. Consider the annulus bounded by $\partial \mathbb{D}$ and the exterior boundary component $\partial_{\mathfrak{e}} S^{\ell, j}$. Extend $\varphi_b | \overline{S^{\ell, j}}$ to this annulus by φ_b . Notice that the restriction of φ_b to this annulus is isotopic to a power of a Dehn twist of the annulus. Then the extended function is an element of $\text{Hom}(\overline{\mathbb{D}}; \partial \mathbb{D} \cup E^{\ell, j} \cup \mathcal{E}^{\ell, j})$ and its mapping class equals $\mathbf{m}_{b(\ell, j)}$. Its image $\mathbf{m}_{b(\ell, j), \infty} = \mathcal{H}_\infty(\mathbf{m}_{b(\ell, j)})$ can be obtained as follows. Extend $\varphi_b | \overline{S^{\ell, j}}$ by the identity to all connected components of $\mathbb{P}^1 \setminus \overline{S^{\ell, j}}$. The class of this extension in $\mathfrak{M}(\mathbb{P}^1; \{\infty\} \cup E^{\ell, j} \cup \mathcal{E}^{\ell, j})$ equals $\mathbf{m}_{b(\ell, j), \infty}$. In other words, put $(\overline{S^{\ell, j}})^c = \mathbb{P}^1$, $\zeta = \mathcal{E}^{\ell, j} \cup \{\infty\}$, and let \mathcal{H}_ζ be the homomorphism

$$(6.34) \quad \mathcal{H}_\zeta : \mathfrak{M}(\overline{S^{\ell, j}}; E^{\ell, j} \cup \partial S^{\ell, j}) \rightarrow \mathfrak{M}(\mathbb{P}^1; E^{\ell, j} \cup \zeta) = \mathfrak{M}((\overline{S^{\ell, j}})^c; E^{\ell, j} \cup \zeta) .$$

(See chapter 2 for notation.) We obtain

$$(6.35) \quad \mathbf{m}_{b(\ell, j), \infty} = \mathcal{H}_\zeta \left(\mathbf{m} \left(\varphi_b | \overline{S^{\ell, j}} \right) \right) .$$

The mappings \mathcal{H}_∂ and \mathcal{H}_ζ are related by an isomorphism I_ζ which acts by conjugation

$$(6.36) \quad \mathcal{H}_\zeta = I_{s_\zeta} \circ \mathcal{H}_\partial .$$

(See chapter 2.) The corresponding arguments apply to $\mathbf{m}_{b(1), \infty}$.

The lemma follows from (6.32), (6.35) and (6.36). \square

The irreducible components of braids. The general case. Let now $b \in \mathcal{B}_n$ be an arbitrary reducible braid, not necessarily pure. Let again φ_b be a representative of the mapping class $\mathbf{m} \in \mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n)$ which is completely reduced by an admissible system \mathcal{C} of curves. Denote by $S^{\ell, j}$ the connected components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ of generation ℓ and by $\text{cyc}^{\ell, i}$ the cycles of the $S^{\ell, j}$ under the mapping φ_b .

The irreducible component $B(1)$ associated to S^1 . Associate a geometric braid to the component S^1 . φ_b fixes S^1 setwise since it fixes the boundary component $\partial \mathbb{D}$. Consider all components $S^{2, j}$, $j = 1, \dots, k'_2$, of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ of second generation. Let $\delta^{2, j} \subset \mathbb{D}$ be the topological disc which is bounded by the exterior boundary $\partial_{\mathfrak{e}} S^{2, j}$ of $S^{2, j}$. Label the φ_b -cycles of the components $S^{2, j}$ by $\text{cyc}^{2, i}$, $i = 1, \dots, k_2$. (Note that $k_2 \leq k'_2$.) Consider a cycle which, after relabeling the $S^{2, j}$, we write as

$$S_1^{2, i} \xrightarrow{\varphi_b} S_2^{2, i} \xrightarrow{\varphi_b} \dots \xrightarrow{\varphi_b} S_{k(2, i)}^{2, i} \xrightarrow{\varphi_b} S_1^{2, i} .$$

Here $k(2, i)$ is the length of the cycle. Take a point $z_1^{2, i} \in S_1^{2, i}$ which is not a distinguished point. For $j = 2, \dots, k(2, i)$, the point $z_j^{2, i} \stackrel{\text{def}}{=} \varphi_b^{j-1}(z_1^{2, i})$ is contained

in $S_j^{2,i} \setminus E_n$. Moreover, $\varphi_b^{j-1}(\partial_{\mathfrak{E}} S_1^{2,i}) = \partial_{\mathfrak{E}} S_j^{2,i}$, $j = 2, \dots, k(2, i)$. Replace φ_b by an isotopic self-homeomorphism of $\mathbb{D} \setminus E_n$ denoted again by φ_b , which has the following property:

$$(6.37) \quad \varphi_b^{k(2,i)}(z_1^{2,i}) = z_1^{2,i}, \quad \varphi_b^{k(2,i)} \text{ fixes } \partial_{\mathfrak{E}} S_1^{2,i} \text{ pointwise.}$$

It can be done so that the new self-homeomorphism is equal to the previous one in a neighbourhood of all points of E_n , in a neighbourhood of all points $z_j^{2,i}$, $j = 1, \dots, k(2, 1) - 1$, and in a neighbourhood of all curves of \mathcal{C} except $\partial_{\mathfrak{E}} S_{k(2,i)}^{2,i}$.

Since $\varphi_b^{k(2,i)} \mid \overline{S_j^{2,i}}$ is conjugate to $\varphi_b^{k(2,i)} \mid \overline{S_1^{2,i}}$, $j = 1, \dots, k(2, i)$, we get

$$(6.38) \quad \varphi_b^{k(2,i)} \text{ fixes each } z_j^{2,i} \text{ and fixes each } \partial_{\mathfrak{E}} S_j^{2,i} \text{ pointwise.}$$

Proceed in the same way with all cycles of generation 2. We obtain an isotopic self-homeomorphism of $\mathbb{D} \setminus E_n$ denoted again by φ , and points $z_j^{2,i}$ so that property (6.38) is satisfied for all i and all $j = 1, \dots, k(2, i)$.

Let as above $\varphi_t \in \text{Hom}(\mathbb{D}; \partial \mathbb{D})$, $t \in [0, 1]$, be a continuous family such that $\varphi_1 = \text{id}$ and $\varphi_0 = \varphi_b$. Hence φ_t is a parametrizing isotopy for a geometric braid representing b . Identify the geometric braid with a continuous mapping $g : [0, 1] \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$. In other words,

$$\varphi_t(E_n) = g(t), \quad t \in [0, 1].$$

Let G be the respective closed geometric braid. Put $E^1 = S^1 \cap E_n$ and let \mathcal{E}^1 be the collection of all points $z_j^{2,i}$, $i = 1, \dots, k_2$, $j = 1, \dots, k(2, i)$. Associate to S^1 the geometric braid

$$(6.39) \quad \varphi_t(E^1 \cup \mathcal{E}^1), \quad t \in [0, 1].$$

Let $B(1)$ be the isotopy class of this geometric braid and $\widehat{B(1)}$ be its conjugacy class. The braid $B(1)$ is equipped with the following additional information. The set of its strands is divided into two subsets E^1 and \mathcal{E}^1 . There is a bijection of the second set \mathcal{E}^1 onto the set of those connected components of generation 2 of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ which are adjacent to S^1 .

Let $n(1)$ be the number of points in $E^1 \cup \mathcal{E}^1$. We write the geometric braid (6.39) as mapping

$$(6.40) \quad g^1 : [0, 1] \rightarrow C_{n(1)}(\mathbb{C})/\mathcal{S}_{n(1)}.$$

See Figure 6.2, where $E^1 = \emptyset$. In this figure the connected components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ of generation 2 are $S_1^{2,1}, S_2^{2,1}, S_3^{2,1}$, which are moved along a 3-cycle by φ_b . The set \mathcal{E}^1 equals $\mathcal{E}^1 = \{z_1^{2,1}, z_2^{2,1}, z_3^{2,1}\}$.

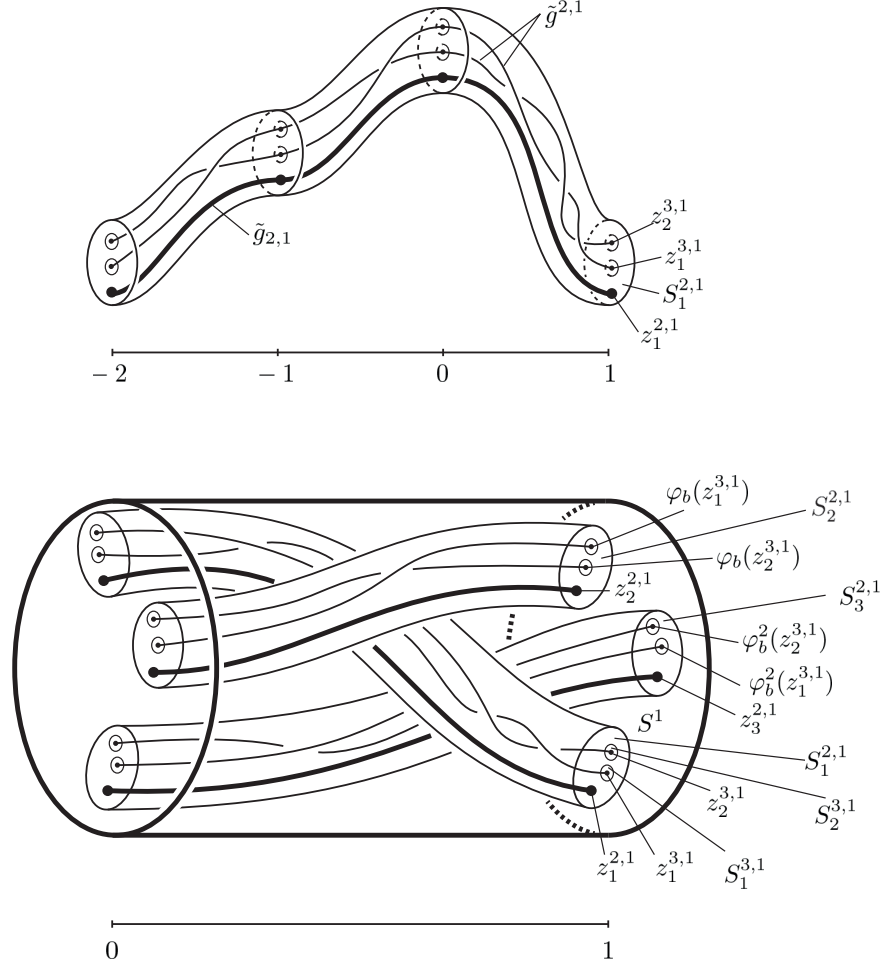


FIGURE 6.2

The closed geometric braid which is associated to the geometric braid (6.40) equals

$$(6.41) \quad G^1(e^{2\pi it}) = g^1(t), \quad t \in [0, 1], \quad G^1 : \partial \mathbb{D} \rightarrow C_{n_1}(\mathbb{C}) / \mathcal{S}_{n(1)}.$$

Denote the mapping class of $B(1)$ in $\mathfrak{M}(\bar{\mathbb{D}}; \partial \mathbb{D}, \mathcal{E}^1 \cup E^1)$ by $\mathbf{m}_{B(1)}$. Consider its image $\mathbf{m}_{B(1), \infty} \stackrel{\text{def}}{=} \mathcal{H}_\infty(\mathbf{m}_{B(1)})$ in $\mathfrak{M}(\mathbb{P}^1; \infty, E^1 \cup \mathcal{E}^1)$ and let $\widehat{\mathbf{m}_{B(1), \infty}}$ be its conjugacy class.

Consider on the other hand the class \mathbf{m}_b of the original braid b and its image $\mathbf{m}_{b, \infty} \stackrel{\text{def}}{=} \mathcal{H}_\infty(\mathbf{m}_b)$ in $\mathfrak{M}(\mathbb{P}^1; \infty, E_n) \cong \mathfrak{M}(\mathbb{C} \setminus E_n)$. Let $\widehat{\mathbf{m}_{b, \infty}}$ be the conjugacy class of $\mathbf{m}_{b, \infty}$. Denote by $\overset{\circ}{\mathbf{m}_{b, \infty}} \mid Y^1$ the irreducible nodal component corresponding to $S^1 \setminus E_n$ which is determined by the system $\mathcal{C} \subset \mathbb{C}$ and the class $\widehat{\mathbf{m}_{b, \infty}}$.

LEMMA 6.3.

$$(6.42) \quad \widehat{\mathring{\mathbf{m}}_{b,\infty} \mid Y^1} = \widehat{\mathbf{m}_{B(1),\infty}}.$$

Sketch of proof of Lemma 6.3. The situation differs from that of Lemma 6.2 by the fact that b is not necessarily a pure braid and, hence, the interior boundary components of S^1 may be permuted by the mapping φ_b . The main modification of the proof consists in the following. Let $(S^1)^{\text{Cl}}$ be the finite point compactification of S^1 , obtained by adding a finite set $Z_\infty^1 \cup \{z_\infty\}$. Here Z_∞^1 contains exactly one point for each interior boundary component of S^1 , and z_∞ corresponds to the exterior boundary component $\partial \mathbb{D}$. We obtain a canonical homomorphism

$$(6.43) \quad \mathcal{H}_\partial : \mathfrak{M}(\overline{S^1}; \partial \mathbb{D}, E^1) \rightarrow \mathfrak{M}((S^1)^{\text{Cl}}; z_\infty, E^1 \cup Z_\infty^1).$$

Notice that the points of the set Z_∞^1 may be permuted by the mappings in the mapping class. The mapping $w : S^1 \rightarrow Y$ induces an isomorphism

$$(6.44) \quad (w^1)^* : \mathfrak{M}((S^1)^{\text{Cl}}; z_\infty, E^1 \cup Z_\infty^1) \rightarrow \mathfrak{M}(\overline{Y^1}; w(z_\infty), w(E^1) \cup (\mathcal{N} \cap \overline{Y^1}))$$

by extension and conjugation of the representatives of the classes. An isomorphism

$$(6.45) \quad \text{Is}_{\mathcal{E}^1 \cup \{\infty\}} = \mathfrak{M}((S^1)^{\text{Cl}}; z_\infty, E^1 \cup Z_\infty^1) \rightarrow \mathfrak{M}(\mathbb{P}^1; \infty, E^1 \cup \mathcal{E}^1)$$

can be defined also for the case when the points of Z_∞^1 may be permuted. The isomorphism acts by conjugation. We obtain a homomorphism

$$(6.46) \quad \mathcal{H}_{\mathcal{E}^1 \cup \{\infty\}} = \text{Is}_{\mathcal{E}^1 \cup \{\infty\}} \circ \mathcal{H}_\partial.$$

The equalities

$$(6.47) \quad \mathring{\mathbf{m}}_{b,\infty} \mid \overline{Y^1} = (w^1)^* \circ \mathcal{H}_\partial(\mathbf{m}(\varphi_b \mid \overline{S^1}))$$

and

$$(6.48) \quad \mathbf{m}_{b(1),\infty} = \mathcal{H}_{\mathcal{E}^1 \cup \{\infty\}}(\mathbf{m}(\varphi_b \mid \overline{S^1}))$$

hold and yield the lemma. The details are left to the reader. \square

The braids $B(\ell, i)$ associated to the φ_b -cycles $\text{cyc}^{\ell, i}$. We will now consider the φ_b -cycles $\text{cyc}^{\ell, i} = (S_1^{\ell, i}, \dots, S_{k(\ell, i)}^{\ell, i})$ of components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$, $\ell \geq 2$, by induction on the number ℓ of their generation. Here $k(\ell, i)$ is the length of the cycle and

$$\varphi_b(S_j^{\ell, i}) = S_{j+1}^{\ell, i}, \quad j = 1, \dots, k(\ell, i) - 1.$$

There is no natural way to associate a geometric braid to the restriction $\varphi_b \mid S_1^{(\ell, i)} \cup \dots \cup S_{k(\ell, i)}^{(\ell, i)}$ of the mapping φ_b . Consider, for instance, a cycle $\text{cyc}^{2, 1}$ of generation 2. The problem is that evaluation of φ_t , $t \in [0, 1]$, on a φ_b -cycle of points contained in the union of the components of $\text{cyc}^{2, 1}$ will contain also information about the braid $B(1)$.

The plan is the following. We will consider instead of the mapping φ_b the $k(\ell, i)$ -th iterate $\varphi_b^{k(\ell, i)}$ and instead of the braid b we consider the iterated braid $b^{k(\ell, i)}$. Associate a geometric braid to the restriction $\varphi_b^{k(\ell, i)} \mid S_1^{\ell, i}$. The isotopy class of this geometric braid will be denoted by $B(\ell, i)$. Its conjugacy class $\widehat{B(\ell, i)}$ can be related to the $k(\ell, i)$ -th iterate of the nodal component $\mathring{\mathbf{m}}_b \mid \widehat{\text{cyc}^{\ell, i}}$. Moreover,

we will show that for the “synthesis” of the conjugacy class \hat{b} it will be sufficient to know the conjugacy class $\widehat{B(1)}$ and all conjugacy classes $\widehat{B(\ell, i)}$ together with additional information.

The mapping $\varphi_b^{k(\ell, i)}$ maps each component $S_j^{\ell, i}$ of the cycle $\text{cyc}^{\ell, i}$ onto itself. We will treat now the mapping $\varphi_b^{k(\ell, i)} \mid S_1^{\ell, i}$ in the same way as we treated the mapping $\varphi_b \mid S^1$, and will see by induction that we may assume that

$$(6.49) \quad \varphi_b^{k(\ell, i)} \text{ fixes } \partial_{\mathfrak{E}} S_1^{\ell, i} \text{ pointwise.}$$

For $\ell = 2$ the relation (6.49) was achieved during the construction of the braid $B(1)$. The statement (6.49) for a given $k(\ell, i)$ implies that the iterate $\varphi_b^{k(\ell, i)}$ permutes the connected components of $\partial_{\mathfrak{J}} S_1^{\ell, i}$ along cycles. Take a $\varphi_b^{k(\ell, i)}$ -cycle of components of $\partial_{\mathfrak{J}} S_1^{\ell, i}$ and the adjacent $\varphi_b^{k(\ell, i)}$ -cycle of components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ of generation $\ell + 1$.

Denote the latter $\varphi_b^{k(\ell, i)}$ -cycle by $\widetilde{\text{cyc}}^{\ell+1, 1} = (S_1^{\ell+1, 1}, \dots, S_{k(\ell+1, 1)}^{\ell+1, 1})$. Here $k(\ell+1, 1)$ is the length of the cycle as a $\varphi_b^{k(\ell, i)}$ -cycle. Notice that there is a corresponding φ_b -cycle of components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ of length $k(\ell, i) \cdot k(\ell+1, 1)$, which we ignore here.

(See Figure 6.2 where the case $(\ell, i) = (2, 1)$ is considered. In the case of the figure $k(2, 1) = 3$. There is only one φ_b^3 -cycle of components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$, namely $\widetilde{\text{cyc}}^{3, 1} = \{S_1^{3, 1}, S_2^{3, 1}\}$ which is of length 2.)

We choose a point $z_1^{\ell+1, 1} \in S_1^{\ell+1, 1} \setminus E_n$. Put $z_j^{\ell+1, 1} = \varphi_b^{k(\ell, i) \cdot (j-1)}(z_1^{\ell+1, 1})$, $j = 2, \dots, k(\ell+1, 1)$. By an isotopy of φ_b which changes φ_b only in a small neighbourhood of $\partial_{\mathfrak{E}} S_{k(\ell+1, 1)}^{\ell+1, 1} \cup \{z_{k(\ell+1, 1)}^{\ell+1, 1}\}$ we may achieve that

$$(6.50) \quad \varphi_b^{k(\ell, i) \cdot k(\ell+1, 1)} \text{ fixes } \partial_{\mathfrak{E}} S_j^{\ell+1, 1} \cup \{z_j^{\ell+1, 1}\} \text{ pointwise for } j = 1, \dots, k(\ell+1, 1).$$

Proceed in the same way with all $\varphi_b^{k(\ell, i)}$ -cycles $\widetilde{\text{cyc}}^{\ell+1, i'}$ of components of $\mathbb{D} \setminus \bigcup_{C \in \mathcal{C}} C$ of generation $\ell + 1$ which are adjacent to $S_1^{\ell, i}$. We see now that condition (6.50) gives the assumption (6.49) for all cycles of generation $\ell + 1$, provided it holds for all cycles of generation not exceeding ℓ . Let $\mathcal{E}^{\ell, i}$ be the collection of all points $z_j^{\ell+1, i'}$ obtained by this procedure. Put $E^{\ell, i} = S_1^{\ell, i} \cap E$.

Let again $\varphi_t \in \text{Hom}(\mathbb{D}; \partial \mathbb{D})$, $t \in [0, 1]$, be a continuous family with $\varphi_0 = \varphi_b$, $\varphi_1 = \text{id}$. Extend this family to a continuous family $\varphi_t \in \text{Hom}(\mathbb{D}; \partial \mathbb{D})$, $t \in [-k(\ell, i) + 1, 1]$, such that $\varphi_{-k(\ell, i)+1} = (\varphi_b)^{k(\ell, i)}$, and, as before $\varphi_1 = \text{id}$. This can be done by putting

$$(6.51) \quad \varphi_t = \varphi_{\tilde{t}} \circ (\varphi_b)^j \text{ for } t = -j + \tilde{t}, \tilde{t} \in [0, 1], j = 0, \dots, k(\ell, i) - 1.$$

This is correctly defined since for $t = -j + 0 = -(j+1) + 1$ the two ways of defining φ_t give the same result.

Notice that this family is a parametrizing isotopy (parametrized by an interval of length $k(\ell, i)$) of the braid $b^{k(\ell, i)}$. In other words, the geometric braid

$$(6.52) \quad \varphi_t(E_n), \quad t \in [-k(\ell, i) + 1, 1],$$

represents $b^{k(\ell,i)}$. (The choice of the interval is due to the fact that the original family φ_t , $t \in [0, 1]$, has the property $\varphi_0 = \varphi_b$, $\varphi_1 = \text{id}$, not vice versa. We followed [22].)

Associate to the cycle $\text{cyc}^{\ell,i}$ the following geometric braid (parametrized by an interval of length $k(\ell, i)$):

$$(6.53) \quad \varphi_t(E^{\ell,i} \cup \mathcal{E}^{\ell,i}), \quad t \in [-k(\ell, i) + 1, 1].$$

(See Figure 6.2 where $\mathcal{E}^{2,1} = \{z_1^{3,1}, z_2^{3,2}\}$, $E^{2,1} = \emptyset$.)

The geometric braid (6.53) induces a permutation of the points of $E^{\ell,i} \cup \mathcal{E}^{\ell,i}$ along cycles. By (6.51) the restriction

$$(6.54) \quad \varphi_t(E^{\ell,i} \cup \mathcal{E}^{\ell,i}), \quad t \in [-j, -j+1], \quad j = 0, \dots, k(\ell, i) - 1,$$

can be written as follows. Put $t = -j + \tilde{t}$, $\tilde{t} \in [0, 1]$. Then

$$(6.55) \quad \varphi_t(E^{\ell,i} \cup \mathcal{E}^{\ell,i}) = \varphi_{\tilde{t}}(\varphi_b^j(E^{\ell,i} \cup \mathcal{E}^{\ell,i})).$$

In other words, we obtain the evaluation of the $\varphi_{\tilde{t}}$, $\tilde{t} \in [0, 1]$, on the set $\varphi_b^j(E^{\ell,i} \cup \mathcal{E}^{\ell,i})$.

Let $\delta_j^{\ell,j}$ be the topological disc in \mathbb{D} which is bounded by the exterior boundary $\partial_{\mathfrak{E}} S_j^{\ell,j}$, $k = 1, \dots, k(\ell, j)$. The set of points

$$(6.56) \quad \bigcup_{j=0}^{k(\ell,i)-1} \varphi_b^j(E^{\ell,i} \cup \mathcal{E}^{\ell,i}), \quad (\varphi_b^0 \stackrel{\text{def}}{=} \text{id}),$$

is contained in $\delta_1^{\ell,i} \cup \delta_2^{\ell,i} \cup \dots \cup \delta_{k(\ell,i)-1}^{\ell,i}$. The points of (6.56) are permuted by φ_b along cycles whose length are multiples of $k(\ell, i)$.

Denote by $n(\ell, i)$ the number of points in $E^{\ell,i} \cup \mathcal{E}^{\ell,i}$. The geometric braid (6.53) can be written as mapping

$$(6.57) \quad \tilde{g}^{\ell,i} : [-k(\ell, i) + 1, 1] \rightarrow C_{n(\ell,i)}(\mathbb{C}) / \mathcal{S}_{n(\ell,i)}.$$

For braids that are not pure it will be more convenient to work with closed geometric braids rather than with geometric braids. Associate to $\tilde{g}^{\ell,i}$ the closed geometric braid

$$(6.58) \quad G^{\ell,i} \left(e^{\frac{2\pi i t}{k(\ell,i)}} \right) = \tilde{g}^{\ell,i}(t), \quad t \in [-k(\ell, i) + 1, 1].$$

Consider the $k(\ell, i)$ -fold covering

$$\widetilde{\partial \mathbb{D}}^{k(\ell,i)} \ni \zeta \rightarrow \zeta^{k(\ell,i)} \in \partial \mathbb{D}$$

of $\partial \mathbb{D}$, and the associated $k(\ell, i)$ -fold covering of the torus $\partial \mathbb{D} \times \mathbb{C}$,

$$(6.59) \quad \widetilde{\partial \mathbb{D}}^{k(\ell,i)} \times \mathbb{C} \ni (\zeta, z) \xrightarrow{p_{k(\ell,i)}} (\zeta^{k(\ell,i)}, z) \in \partial \mathbb{D} \times \mathbb{C}.$$

We may consider $G^{\ell,i}$ as closed geometric braid in the $k(\ell, i)$ -fold covering $\widetilde{\partial \mathbb{D}}^{k(\ell,i)} \times \mathbb{C}$ of $\partial \mathbb{D} \times \mathbb{C}$,

$$(6.60) \quad \zeta \rightarrow G^{\ell,i}(\zeta) = G^{\ell,i} \left(e^{\frac{2\pi i t}{k(\ell,i)}} \right) \in C_{n(\ell,i)}(\mathbb{C}) / \mathcal{S}_{n(\ell,i)}, \quad \zeta \in \widetilde{\partial \mathbb{D}}^{k(\ell,i)},$$

where $\zeta = e^{\frac{2\pi i t}{k(\ell,i)}}$, $t \in [-k(\ell, i) + 1, 1]$.

Reparametrize the geometric braid $\tilde{g}^{\ell,i}$ to obtain a mapping defined on the unit interval, in other words, put

$$g^{\ell,i}(t) = \tilde{g}^{\ell,i}(k(\ell, i)(t - 1) + 1) : [0, 1] \rightarrow C_{n(\ell,i)}(\mathbb{C}) / \mathcal{S}_{n(\ell,i)}.$$

The isotopy class of the geometric braid $g^{\ell,i}$ is denoted by $B(\ell, i)$ and its conjugacy class is denoted by $\widehat{B(\ell, i)}$. The conjugacy class $\widehat{B(\ell, i)}$ can be identified with the free isotopy class of the closed braid $G^{\ell,i}$ in $\widetilde{\partial \mathbb{D}}^{k(\ell,i)} \times \mathbb{C}$.

The conjugacy class $\widehat{B(\ell, i)}$ is equipped with the following additional information. There is a number $k(\ell, i)$ associated to $\widehat{B(\ell, i)}$, which is equal to the length of the φ_b -cycle $\text{cyc}^{\ell,i}$ of connected components of $\mathbb{D} \setminus \cup_{\mathcal{C}} C$ of generation $\ell + 1$ corresponding to the class $\widehat{B(\ell, i)}$. The set of strands of the representing braids are divided into two sets, $E^{\ell,i}$ and $\mathcal{E}^{\ell,i}$. There is a bijection of the second set onto the set of those connected components of generation $\ell + 1$ of $\mathbb{D} \setminus \cup_{\mathcal{C}} C$ which are adjacent to a single connected component of the cycle $\text{cyc}^{\ell,i}$.

We described the “analysis” for not necessarily pure braids.

We will now describe the “synthesis”. It will be convenient to do this in terms of closed braids.

We make the following identification for closed geometric braids. Let $F : \partial \mathbb{D} \rightarrow C_n(\mathbb{C})/\mathcal{S}_n$ be a closed geometric braid. Interpret for each $\zeta \in \partial \mathbb{D}$ the unordered n -tuple $F(\zeta)$ as a subset of \mathbb{C} . The set

$$(6.61) \quad \gamma(F) = \{(\zeta, F(\zeta)) : \zeta \in \partial \mathbb{D}\} \subset \partial \mathbb{D} \times \mathbb{C}$$

is a closed oriented one-dimensional submanifold of $\partial \mathbb{D} \times \mathbb{C}$. It has the following property. Denote by $\mathcal{P}_{\partial \mathbb{D}} : \partial \mathbb{D} \times \mathbb{C} \rightarrow \partial \mathbb{D}$ the canonical projection. The restriction $\mathcal{P}_{\partial \mathbb{D}}|_{\gamma(F)}$ is an orientation-preserving covering of $\partial \mathbb{D}$. We identify closed geometric braids with closed one-dimensional submanifolds of $\partial \mathbb{D} \times \mathbb{C}$ having this property.

Suppose now that the conjugacy classes $\widehat{B(1)}$ and $\widehat{B(\ell, i)}$ are given together with the additional information. Start with the conjugacy class $\widehat{B(1)}$. Represent it by a geometric braid f^1 . Denote its base point by $E_f^1 \cup \mathcal{E}_f^1$. Denote by F^1 the corresponding closed geometric braid. The set $\gamma(F^1)$ intersects the fiber $\{1\} \times \mathbb{C}$ along the set $E_f^1 \cup \mathcal{E}_f^1$. The set \mathcal{E}_f^1 corresponds to the set of connected components of $\mathbb{D} \setminus \cup_{\mathcal{C}} C$ of generation 2. They are permuted along cycles $\text{cyc}^{2,i}$ by the permutation induced by the geometric braid f^1 . Consider a respective cycle of points $(z_1^{2,1}(f), z_2^{2,1}(f), \dots, z_{k(2,1)}^{2,1}(f))$ in \mathcal{E}_f^1 . Denote by $\gamma_{2,1}(F)$ the connected component of $\gamma^1(F)$ which corresponds to this cycle (i.e. which intersects the fiber $\{1\} \times \mathbb{C}$ along the points of the cycle). We have

$$(6.62) \quad \gamma_{2,1}(F) = \left\{ \left(e^{2\pi i t}, \varphi_t \left(\{z_1^{2,1}(f), z_2^{2,1}(f), \dots, z_{k(2,1)}^{2,1}(f)\} \right) \right) : t \in [0, 1] \right\}.$$

Consider the projection $\partial \mathbb{D} \times \mathbb{C} \rightarrow \partial \mathbb{D}$. The curve $\gamma_{2,1}(F)$ covers $\partial \mathbb{D}$ $k(2, 1)$ times. Consider the covering (6.59) and the lift $\widetilde{\gamma_{2,1}(F)} \subset \widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$ of $\gamma_{2,1}(F) \subset \partial \mathbb{D} \times \mathbb{C}$, which passes through the point $(1, z_1^{2,1}(f)) \in \widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$. The lift $\widetilde{\gamma_{2,1}(F)}$ is a closed curve which covers $\widetilde{\partial \mathbb{D}}^{k(2,1)}$ once. We write

$$(6.63) \quad \widetilde{\gamma_{2,1}(F)} = \left\{ (\zeta, F_{2,1}(\zeta)) : \zeta \in \widetilde{\partial \mathbb{D}}^{k(2,1)} \right\} \subset \widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C},$$

where $F_{2,1} : \widetilde{\partial \mathbb{D}}^{k(2,1)} \rightarrow \mathbb{C}$ is a continuous function defined by $\widetilde{\gamma_{2,1}(F)}$ (See also Figure 6.2).

Consider now the conjugacy class $\widehat{B(2,1)}$ and the natural number $k(2,1)$ associated to it, which is equal to the length of the φ_b -cycle $\text{cyc}^{2,1}$ of connected components of $\mathbb{D} \setminus \cup_C C$ corresponding to $\widehat{B(2,1)}$. Take a representing geometric braid $\tilde{f}^{2,1} : [-k(2,1) + 1, 1] \rightarrow \mathcal{C}_{n(2,1)}(\mathbb{C})/\mathcal{S}_{n(2,1)}$. Let $E_f^{2,1} \cup \mathcal{E}_f^{2,1}$ be the base point of the representing geometric braid, where $\mathcal{E}_f^{2,1}$ is in bijective correspondence with the set of connected components of generation 3 of $\mathbb{D} \setminus \cup_C C$ which are adjacent to the component $S_1^{2,1}$ of the cycle $\text{cyc}^{2,1}$.

Let $F^{2,1}$ be the closed geometric braid in the $k(2,1)$ -fold covering $\widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$ of $\partial \mathbb{D} \times \mathbb{C}$, associated to $\tilde{f}^{2,1}$.

For an unordered n -tuple of points $\{z_1, z_2, \dots, z_n\} \in \mathcal{C}_n(\mathbb{C})/\mathcal{S}_n$, a positive number ϵ and a number $z \in \mathbb{C}$ we put

$$z \boxplus \epsilon \{z_1, z_2, \dots, z_n\} \stackrel{\text{def}}{=} \{z + \epsilon z_1, z + \epsilon z_2, \dots, z + \epsilon z_n\} \in \mathcal{C}_n(\mathbb{C})/\mathcal{S}_n.$$

Let $n(2,1)$ be the number of points in $E_f^1 \cup \mathcal{E}_f^1$. We define the mapping

$$F_{2,1} \boxplus \epsilon^{2,1} F^{2,1} : \widetilde{\partial \mathbb{D}}^{k(2,1)} \rightarrow \mathcal{C}_{n(2,1)}/\mathcal{S}_{n(2,1)},$$

to be equal to

$$F_{2,1}(\zeta) \boxplus \epsilon^{2,1} F^{2,1}(\zeta) \in \mathcal{C}_{n(2,1)}/\mathcal{S}_{n(2,1)},$$

for $\zeta \in \widetilde{\partial \mathbb{D}}^{k(2,1)}$. Denote by $\gamma^{2,1}(F)$ the set

$$(6.64) \quad \gamma^{2,1}(F) \stackrel{\text{def}}{=} \left\{ (\zeta, F_{2,1}(\zeta) \boxplus \epsilon^{2,1} F^{2,1}(\zeta)) : \zeta \in \widetilde{\partial \mathbb{D}}^{k(2,1)} \right\} \subset \widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}.$$

If $\epsilon^{2,1}$ is small enough the obtained subset $\gamma^{2,1}(F)$ of $\widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$ is contained in a small tubular neighbourhood of $\gamma_{2,1}(F)$.

The projection $\gamma^{2,1}(F) \stackrel{\text{def}}{=} p_{k(2,1)}(\gamma^{2,1}(F))$ defines a closed geometric braid in $\partial \mathbb{D} \times \mathbb{C}$. We denote this closed geometric braid by

$$(6.65) \quad F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}.$$

Recall that the number of strands of the closed geometric braid $F^{2,1}$ equals $n(2,1)$. The number of strands of (6.65) equals $k(2,1)n(2,1)$.

Replace now the connected component $\gamma_{2,1}(F) = p_{k(2,1)}(\gamma_{2,1}(F)) \subset \partial \mathbb{D} \times \mathbb{C}$ of $\gamma^1(F)$ by $\gamma^{2,1}(F) \subset \partial \mathbb{D} \times \mathbb{C}$. If $\epsilon^{2,1}$ is small enough the obtained subset of $\partial \mathbb{D} \times \mathbb{C}$ can be identified with a closed geometric braid which is denoted by

$$(6.66) \quad F^1 \sqcup (F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}).$$

The respective geometric braid is denoted by

$$(6.67) \quad f^1 \sqcup (f_{2,1} \boxplus \epsilon^{2,1} f^{2,1}).$$

The number of strands of (6.66) equals $n(1) + k(2,1)(n(2,1) - 1)$. Denote by $B(1) \sqcup B(2,1)$ the braid represented by (6.67) and by $\widehat{B(1)} \sqcup \widehat{B(2,1)}$ its conjugacy class. Note that $\widehat{B(1)} \sqcup \widehat{B(2,1)}$ is the free isotopy class of closed braids represented by (6.66).

The class $B(1) \sqcup B(2,1)$ is related to the above chosen mapping class φ_b and the continuous family $\varphi_t \in \text{Hom}^+(\mathbb{D}; \partial \mathbb{D})$ as follows. Consider the closed geometric braid G^1 (see (6.39), (6.40) and (6.41)) in $\partial \mathbb{D} \times \mathbb{C}$ and the geometric braid $\tilde{G}^{2,1}$ in

$\widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$ (see (6.53), (6.57), and (6.58)). Do the same operation of composition as in (6.62), (6.63), (6.64), (6.66) and (6.67), with F^1 replaced by G^1 , $F_{2,1}$ replaced by $G_{2,1}$ (which is defined using the parametrizing isotopy φ_t in the same way as for defining G^1), and $F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}$ replaced by $G^{2,1}$. We obtain a closed geometric braid in $\partial \mathbb{D} \times \mathbb{C}$ denoted by $G^1 \sqcup G^{2,1}$. The respective geometric braid is denoted by $g^1 \sqcup g^{2,1}$.

In terms of the parametrizing isotopy φ_t , $t \in [0, 1]$, for φ_b we have
(6.68)

$$(G^1 \sqcup G^{2,1})(e^{2\pi i t}) = \varphi_t \left(E' \cup E'' \cup \varphi_b(E'') \cup \dots \cup \varphi_b^{k(2,1)-1}(E'') \right), \quad t \in [0, 1],$$

where

$$(6.69) \quad E' = \mathcal{E}^1 \setminus \{z_1^{2,1}, \dots, z_{k(2,1)}^{2,1}\}, \text{ and } E'' = E^{2,1} \cup \mathcal{E}^{2,1}.$$

(See Figure 6.2.)

The operation $G^1 \sqcup G^{2,1}$ can be intuitively interpreted as follows. Consider the closed geometric braid $G^1(e^{2\pi i t}) = \varphi_t(E^1 \cup \mathcal{E}^1)$, $t \in [0, 1]$. Let $\delta_j^{2,1} \subset \mathbb{D}$ be the topological discs bounded by the exterior boundary components of the sets $S_j^{2,1}$ of the cycle $\text{cyc}^{2,1}$, labeled so that $z_j^{2,1} \in \delta_j^{2,1}$. Consider the “partially thickened” closed geometric braid

$$(6.70) \quad \mathbf{G}^1(e^{2\pi i t}) = \varphi_t \left(E^1 \cup \left(\mathcal{E}^1 \setminus \{z_1^{2,1}, \dots, z_{k(2,1)}^{2,1}\} \right) \cup \bigcup_{j=1}^{k(2,1)} \overline{\delta_j^{2,1}} \right), \quad t \in [0, 1].$$

\mathbf{G}^1 is obtained from G^1 by surrounding the connected component $\gamma_{2,1}$ of the closed braid G^1 by a solid braided torus $T_{2,1}$. Lift $T_{2,1}$ to a solid torus $\widetilde{T}_{2,1} \subset \widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$ which surrounds $\widetilde{\gamma_{2,1}}$. Trivialize $\widetilde{T}_{2,1}$ by the inclusion into the product $\widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$. Insert the closed geometric braid $\tilde{G}^{2,1}$ into the solid torus $\widetilde{T}_{2,1}$. We obtain a closed braid in $\widetilde{\partial \mathbb{D}}^{k(2,1)} \times \mathbb{C}$. Take its projection $p_{k(2,1)}$ to $\partial \mathbb{D} \times \mathbb{C}$. We obtain a closed braid in $\partial \mathbb{D} \times \mathbb{C}$ which is inserted into $T_{2,1}$. Replace in (6.70) the braided torus defined by $\varphi_t \left(\bigcup_j \overline{\delta_j^{2,1}} \right)$ by this closed geometric braid. The thus obtained closed geometric braid in $\partial \mathbb{D} \times \mathbb{C}$ is the closed geometric braid $G^1 \sqcup G^{2,1}$.

This interpretation suggests the following fact. Consider a free isotopy $G^1(s)$, $s \in [0, 1]$, of closed geometric braids with $G^1(0) = G^1$, and a free isotopy of closed geometric braids $G^{2,1}(s)$, $s \in [0, 1]$, with $G^{2,1}(0) = G^{2,1}$ so that for all $s \in [0, 1]$ the described operation $G^1(s) \sqcup G^{2,1}(s)$ gives a closed geometric braid. Then all closed geometric braids $G^1(s) \sqcup G^{2,1}(s)$, $s \in [0, 1]$, are free isotopic.

Indeed, we have the following lemma.

LEMMA 6.4. *For $\epsilon^{2,1} > 0$ small enough the closed geometric braids $G^1 \sqcup G^{2,1}$ and $F^1 \sqcup (F_{2,1} \boxplus \epsilon^{2,1} F^{2,1})$ are free isotopic.*

Sketch of proof of Lemma 6.4. A proof can be given along the same lines as the proof of Lemma 6.1.

Consider the closed geometric braids G^1 and F^1 and the corresponding geometric braids g^1 and f^1 . Since g^1 and f^1 are free isotopic, by Proposition 2.4 there exists a continuous family $\varphi_t^s \in \text{Hom}^+(\mathbb{D}; \partial \mathbb{D})$, $(t, s) \in [0, 1] \times [0, 1]$, with the following properties. For the family φ_t used in (6.12) we have $\varphi_t^0 = \varphi_t$. There is

a continuous family $\psi^s \in \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D})$, $s \in [0, 1]$, with $\psi^0 = \text{id}$, so that for each $s \in [0, 1]$ the evaluation

$$\varphi_t^s(\psi^s(E^1 \cup \mathcal{E}^1)), \quad t \in [0, 1],$$

defines a geometric braid with base point $\psi^s(E^1 \cup \mathcal{E}^1)$ (see the notation of Remark 2.1). The thus defined family $f^1(s)$, $s \in [0, 1]$, is a free isotopy of geometric braids joining g^1 and f^1 . Moreover, the family of evaluations on the whole set E_n , given for each $s \in [0, 1]$ by $\varphi_t^s(\psi^s(E_n))$, $t \in [0, 1]$, defines a free isotopy of geometric braids. It joins the original geometric braid $\varphi_t(E_n) = \varphi_t^0(E_n)$, $t \in [0, 1]$, with base point E_n with the geometric braid

$$\varphi_t^1(\psi^1(E_n)), \quad t \in [0, 1],$$

with base point $\psi^1(E_n)$, which is obtained for $s = 1$.

Replace all objects related to the base point E_n , and those related to its subsets, as well as those related to the chosen topological discs, by their images under ψ^1 . Doing so we may assume that $g^1 = f^1$. Then also for the connected components of $\gamma^1(F) = \gamma^1(G)$ we have $\gamma_{2,1}(F) = \gamma_{2,1}(G)$, and for the related functions on $\widetilde{\partial\mathbb{D}}^{k(2,1)}$ we have $F_{2,1} = G_{2,1}$.

Consider now the closed geometric braids $F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}$ and $G^{2,1}$ in $\widetilde{\partial\mathbb{D}}^{k(2,1)} \times \mathbb{C}$. They are free isotopic and both are contained in the tubular neighbourhood

$$(6.71) \quad \bigcup_{t \in [-k(2,1)+1, 1]} \{e^{\frac{2\pi i t}{k(2,1)}}\} \times \varphi_t(\delta^{2,1})$$

of the graph of $G_{2,1}$. (See (6.51) for the definition of φ_t on $[-k(2,1)+1, 1]$.) Hence, there is a free isotopy $G^{2,1}(s)$, $s \in [0, 1]$, of closed geometric braids in $\widetilde{\partial\mathbb{D}}^{k(2,1)} \times \mathbb{C}$ with $G^{2,1}(0) = G^{2,1}$, $G^{2,1}(1) = F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}$, such that all closed geometric braids of the isotopy are contained in the cylinder (6.71).

Take the 1-dimensional submanifolds of $\widetilde{\partial\mathbb{D}}^{k(2,1)} \times \mathbb{C}$ defined by $F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}$, and by $G^{2,1}$, respectively. Consider their projections $\gamma^{2,1}(F)$ and $\gamma^{2,1}(G)$ to $\partial\mathbb{D} \times \mathbb{C}$. In the same way we consider the one-dimensional submanifolds $\widetilde{\gamma_s^{2,1}}$ of $\widetilde{\partial\mathbb{D}}^{k(2,1)} \times \mathbb{C}$ defined by the isotopy $G^{2,1}(s)$, $s \in [0, 1]$, and let $\gamma_s^{2,1}$ be their projections to $\partial\mathbb{D} \times \mathbb{C}$.

Replace the connected component $\gamma_{2,1}(F) = \gamma_{2,1}(G)$ of $\gamma^1(F) = \gamma^1(G)$ by $\gamma_s^{2,1}$, $s \in [0, 1]$. We obtain a free isotopy of closed braids in $\partial\mathbb{D} \times \mathbb{C}$ joining $G^1 \sqcup G^{2,1}$ and $F^1 \sqcup (F_{2,1} \boxplus \epsilon^{2,1} F^{2,1})$. The details are left to the reader. \square

The inductive construction goes as follows. Suppose for some generation ℓ and a number $i < k_\ell$ we know the conjugacy classes $\widehat{B(\ell, i)}$ and

$$(6.72) \quad \widehat{B(1)} \sqcup \widehat{B(2, 1)} \sqcup \dots \sqcup \widehat{B(\ell-1, k_{\ell-1})} \sqcup \dots \sqcup \widehat{B(\ell, i-1)}.$$

Assume by induction that a representative

$$(6.73) \quad F^1 \sqcup (F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}) \sqcup \dots \sqcup (F_{\ell-1, k_{\ell-1}} \boxplus \epsilon^{\ell-1, k_{\ell-1}} F^{\ell-1, k_{\ell-1}}) \sqcup \dots \sqcup (F_{\ell, i-1} \boxplus \epsilon^{\ell, i-1} F^{\ell, i-1})$$

is given together with a free isotopy joining the closed geometric braid (6.73) and the closed geometric braid

$$(6.74) \quad G^1 \sqcup G^{2,1} \sqcup \dots \sqcup G^{\ell-1, k_{\ell-1}} \sqcup \dots \sqcup G^{\ell, i-1}.$$

Here the closed geometric braids $G^{\ell', i'}$ are related to the closed geometric braid G by (6.53), (6.57), and (6.58). For instance if $\ell = 2$ then the geometric braid

$$(6.75) \quad g^1 \sqcup g^{2,1} \sqcup \dots \sqcup g^{2,i-1}$$

related to (6.74) is equal to

$$(6.76) \quad \varphi_t(E^1 \cup \bigcup_{j=1}^{i-1} E'_j), \quad t \in [0, 1],$$

where

$$(6.77) \quad E'_j = (E^{2,j} \cup \mathcal{E}^{2,j}) \cup \varphi_b(E^{2,j} \cup \mathcal{E}^{2,j}) \cup \dots \cup \varphi_b^{k(2,j)-1}(E^{2,j} \cup \mathcal{E}^{2,j}).$$

Applying Proposition 2.4, we may assume that the closed geometric braids (6.73) and (6.74) are equal.

Consider the conjugacy class $\widehat{B(\ell, i)}$. It corresponds to a cycle $\text{cyc}^{\ell, i}$ of generation ℓ and length $k(\ell, i)$. Represent $\widehat{B(\ell, i)}$ by a closed geometric braid $F^{\ell, i}$ in $\widetilde{\partial \mathbb{D}}^{k(\ell, i)} \times \mathbb{C}$,

$$(6.78) \quad F^{\ell, i} : \widetilde{\partial \mathbb{D}}^{k(\ell, i)} \rightarrow C_{n(\ell, i)}(\mathbb{C}) / \mathcal{S}_{n(\ell, i)}.$$

Identify $F^{\ell, i}$ with a closed one-dimensional submanifold $\widetilde{\gamma^{\ell, i}(F)}$ of $\widetilde{\partial \mathbb{D}}^{k(\ell, i)} \times \mathbb{C}$ and take its projection $\gamma^{\ell, i}(F)$ to $\partial \mathbb{D} \times \mathbb{C}$. Identify (6.73) with a 1-dimensional submanifold of $\partial \mathbb{D} \times \mathbb{C}$. Let $\gamma_{\ell, i}(F)$ be the connected component of this one-dimensional submanifold which corresponds to the cycle $\text{cyc}^{\ell, i}$. Surround $\gamma_{\ell, i}(F)$ by a closed torus $T_{\ell, i}$ which is embedded into $\partial \mathbb{D} \times \mathbb{C}$ and is trivialized by the embedding. Insert a free isotopic copy of $\widetilde{\gamma^{\ell, i}(F)}$ into $T_{\ell, i}$.

More formally, let $\widetilde{\gamma_{\ell, i}(F)}$ be a lift of $\gamma_{\ell, i}(F)$ to $\widetilde{\partial \mathbb{D}}^{k(\ell, i)} \times \mathbb{C}$. $\widetilde{\gamma_{\ell, i}(F)}$ is the graph of a function $F_{\ell, i}$ over $\widetilde{\partial \mathbb{D}}^{k(\ell, i)}$. Take a sufficiently small positive number $\epsilon^{\ell, i}$ and replace $F_{\ell, i}$ by the closed geometric braid

$$(6.79) \quad F_{\ell, i} \boxplus \epsilon^{\ell, i} F^{\ell, i} : \widetilde{\partial \mathbb{D}}^{k(\ell, i)} \rightarrow C_{n(\ell, i)}(\mathbb{C}) / \mathcal{S}_{n(\ell, i)}.$$

Identify the closed braid $F_{\ell, i} \boxplus \epsilon^{\ell, i} F^{\ell, i}$ in $\widetilde{\partial \mathbb{D}}^{k(\ell, i)} \times \mathbb{C}$ with a closed one-dimensional submanifold of $\widetilde{\partial \mathbb{D}}^{k(\ell, i)} \times \mathbb{C}$ and let $\gamma^{\ell, i}(F)$ be its projection to $\partial \mathbb{D} \times \mathbb{C}$. For the closed geometric braid (6.73) in $\partial \mathbb{D} \times \mathbb{C}$ we consider the following operation. Replace the connected component $\gamma_{\ell, i}(F)$ of (6.73) (considered as submanifold of $\partial \mathbb{D} \times \mathbb{C}$) by $\gamma^{\ell, i}(F)$. We obtain a new one-dimensional submanifold of $\partial \mathbb{D} \times \mathbb{C}$ which defines a closed geometric braid denoted by

$$(6.80) \quad F^1 \sqcup (F_{2,1} \boxplus \epsilon^{2,1} F^{2,1}) \sqcup \dots \sqcup (F_{\ell-1, k_{\ell-1}} \boxplus \epsilon^{\ell-1, k_{\ell-1}} F^{\ell-1, k_{\ell-1}}) \sqcup \dots \sqcup (F_{\ell, i-1} \boxplus \epsilon^{\ell, i-1} F^{\ell, i-1}) \sqcup (F_{\ell, i} \boxplus \epsilon^{\ell, i} F^{\ell, i}).$$

The analog of Lemma 6.4 holds: the closed geometric braid (6.80) is free isotopic to

$$(6.81) \quad G^1 \sqcup G^{2,1} \sqcup \dots \sqcup G^{\ell-1, k_{\ell-1}} \sqcup \dots \sqcup G^{\ell, i-1} \sqcup G^{\ell, i}.$$

Denote the free isotopy class of (6.80) (which is also equal to the free isotopy class of (6.81)) by

$$(6.82) \quad \widehat{B(1)} \sqcup \widehat{B(2,1)} \dots \sqcup \widehat{B(\ell-1, k_{\ell-1})} \sqcup \dots \sqcup \widehat{B(\ell, i-1)} \sqcup \widehat{B(\ell, i)}.$$

The geometric braid related to (6.81) is equal to

$$(6.83) \quad \varphi_t(E'(\ell, i) \cup \mathcal{E}'(\ell, i)), \quad t \in [0, 1],$$

for subsets $E'(\ell, i)$ and $\mathcal{E}'(\ell, i)$ of E_n . The set $E'(\ell, i) \cup \mathcal{E}'(\ell, i)$ is obtained from $E'(\ell, i-1) \cup \mathcal{E}'(\ell, i-1)$ as follows. Replace a φ_b -cycle of points in $\mathcal{E}'(\ell, i-1)$ by the respective φ_b -orbit of the set $E^{\ell, i} \cup \mathcal{E}^{\ell, i}$. Divide the obtained set into the subsets $E'(\ell, i)$ and $\mathcal{E}'(\ell, i)$, where $\mathcal{E}'(\ell, i)$ consists of the points which correspond to an inner boundary component of a connected component of $\mathbb{D} \setminus \cup_{\mathcal{C}} C$ of generation $\ell-1$.

If $i = k_\ell$ we consider the class $\widehat{B(\ell+1, 1)}$ and find in the same way as above a closed geometric braid which is isotopic to

$$(6.84) \quad G^1 \sqcup G^{2,1} \sqcup \dots \sqcup G^{\ell, k_\ell} \sqcup G^{\ell+1, 1}.$$

The represented conjugacy class is denoted by

$$(6.85) \quad \widehat{B(1)} \sqcup \widehat{B(2,1)} \dots \sqcup \widehat{B(\ell, k_\ell)} \sqcup \widehat{B(\ell+1, 1)}.$$

The geometric braid corresponding to (6.84) is equal to

$$(6.86) \quad \varphi_t(E'(\ell+1, 1) \cup \mathcal{E}'(\ell+1, 1)), \quad t \in [0, 1],$$

for subsets $E'(\ell+1, 1)$ and $\mathcal{E}'(\ell+1, 1)$ of E_n which can be obtained by induction similarly as above.

Taking into account that for the last generation N we have $\mathcal{E}^{N, i} = \emptyset$ for $i = 1, \dots, k_N$, we see that

$$(6.87) \quad G = G^1 \sqcup \dots \sqcup G^{N, 1} \sqcup \dots \sqcup G^{N, k_N},$$

and, hence

$$(6.88) \quad \hat{b} = \widehat{B(1)} \sqcup \dots \sqcup \widehat{B(N, 1)} \sqcup \dots \sqcup \widehat{B(N, k_N)}.$$

This finishes the “synthesis” for braids.

We relate now the braids $B(\ell, i)$ to nodal mapping classes and describe the “synthesis” for mapping classes.

Denote by $\mathbf{m}_{B(\ell, i)}$ the mapping class of $B(\ell, i)$ in $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E^{\ell, i} \cup \mathcal{E}^{\ell, i})$, put $\mathbf{m}_{B(\ell, i), \infty} = \mathcal{H}_\infty(\mathbf{m}_{B(\ell, i)})$, and let $\widehat{\mathbf{m}_{B(\ell, i), \infty}}$ be the respective conjugacy class. Also, put $\mathbf{m}_{b^{k(\ell, i)}, \infty} = \mathcal{H}_\infty(\mathbf{m}_{b^{k(\ell, i)}})$.

We considered a system \mathcal{C} of curves which reduces the representing homeomorphism φ_b of the braid b . The system \mathcal{C} also reduces a representing homeomorphism for $b^{k(\ell, i)}$ (though it may happen that it does not completely reduce $b^{k(\ell, i)}$). With the previous choice of the nodal surface Y and the surjection w we denote by $\mathring{\mathbf{m}}_{b^{k(\ell, i)}, \infty}$ the mapping class on Y induced by the mapping class $\mathbf{m}_{b^{k(\ell, i)}, \infty}$ of the braid $b^{k(\ell, i)}$.

The following lemma holds and is proved in the same way as Lemma 6.2.

LEMMA 6.5.

$$\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty}} \mid Y_1^{\ell,i} = \widehat{\mathbf{m}_{B(\ell,i),\infty}}.$$

We will come now to the “synthesis” for mapping classes. We will recover the conjugacy class $\widehat{\mathring{\mathbf{m}}_{b,\infty}}$ of nodal mappings from knowing for each nodal cycle $\mathring{\text{cyc}}^{\ell,i}$ the conjugacy class $\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty}} \mid Y_1^{\ell,i}$ of the restriction of $\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty}$ to a component $Y_1^{\ell,i}$ of the cycle. The key point is the following Lemma on conjugation. It was known and used by Bers [5] but is not worked out in detail in [5].

LEMMA 6.6. (**Lemma on conjugation**) *Let Q_1, \dots, Q_k be topological spaces. Let φ be a self-homeomorphism of (the formal disjoint union) $\bigcup_{j=1}^k Q_j$ which permutes the Q_j cyclically, i.e. $\varphi(Q_j) = Q_{j+1}$, $j = 1, \dots, k-1$, $\varphi(Q_k) = Q_1$. Let ψ be another self-homeomorphism of $\bigcup_{j=1}^k Q_j$. Suppose that $\psi(Q_j) = Q_{j+1}$, $j = 1, \dots, k-1$, $\psi(Q_k) = Q_1$, and $\psi^k \mid Q_1 = \varphi^k \mid Q_1$. Then φ and ψ are conjugate, i.e. there exists a self-homeomorphism $\sigma : \bigcup_{j=1}^k Q_j \rightarrow \bigcup_{j=1}^k Q_j$ which fixes each Q_j setwise and such that $\psi = \sigma^{-1} \circ \varphi \circ \sigma$.*

Suppose for each j there is a subset $Q'_j \subset Q_j$, such that $\psi \mid Q'_j = \varphi \mid Q'_j$ for each j . Then σ may be chosen to fix Q'_j pointwise.

medskip

COROLLARY 6.1. *Let Q be an oriented manifold. Suppose Q_1, \dots, Q_k are disjoint open subsets of Q . Put $Q_0 = Q \setminus \bigcup_{j=1}^k Q_j$. Suppose φ and ψ are self-homeomorphisms of Q such the following conditions hold.*

•

$$(6.89) \quad \varphi \mid \bar{Q}_0 = \chi_0 \circ (\psi \mid \bar{Q}_0)$$

for a self-homeomorphism χ_0 of \bar{Q}_0 which is isotopic to the identity through homeomorphisms in $\text{Hom}^+(\bar{Q}_0; \bigcup_{j=1}^k \partial Q_j)$.

- φ and ψ permute the Q_j , $j = 1, \dots, k$, along the same cycle:

$$\begin{aligned} \varphi(Q_j) &= Q_{j+1}; \quad \psi(Q_j) = Q_{j+1}, \quad j = 1, \dots, k-1, \\ \varphi(Q_k) &= Q_1; \quad \psi(Q_k) = Q_1. \end{aligned}$$

•

$$(6.90) \quad \varphi^k \mid \bar{Q}_1 = \chi_1 \circ (\psi^k \mid \bar{Q}_1)$$

for a self-homeomorphism χ_1 of \bar{Q}_1 which is isotopic to the identity through homeomorphisms in $\text{Hom}^+(\bar{Q}_1; \partial Q_1)$.

Then φ is obtained from ψ by isotopy through homeomorphisms in $\text{Hom}^+ \left(Q; \bigcup_{j=1}^k \partial Q_j \right)$ followed by conjugating with a self-homeomorphism of Q which fixes each Q_j setwise and the union $\bigcup_{j=1}^k \partial Q_j$ pointwise. In other words, φ and ψ are in the same conjugacy class in $\hat{\mathfrak{M}}(Q; \bigcup_{j=1}^k \partial Q_j)$.

We postpone the proofs. The following lemma shows that the conjugacy class $\widehat{\mathfrak{m}_{b,\infty} \mid \text{cyc}^{\ell,i}}$ is determined by the conjugacy class $\widehat{\mathfrak{m}_{b^{k(\ell,i)},\infty} \mid Y_1^{\ell,i}}$. Moreover, the proof of the lemma describes the representatives of the class. We keep to the admissible system of curves \mathcal{C} , to the reference nodal surface Y , the reference surjection $w : \overline{\mathbb{D}} \rightarrow Y$ and the notation $\text{cyc}^{\ell,i}$ for the nodal cycles on Y . The following lemma holds.

LEMMA 6.7. *The following equality holds for conjugacy classes of mappings*

$$(6.91) \quad \left\{ \widehat{\psi \in \mathfrak{m}_{b^{k(\ell,i)},\infty} \mid Y_1^{\ell,i}} \right\} = \left\{ (\overset{\circ}{\varphi})^{k(\ell,i)} \mid Y_1^{\ell,i} : \overset{\circ}{\varphi} \in \widehat{\mathfrak{m}_{b,\infty} \mid \text{cyc}^{\ell,i}} \right\}.$$

Proof. It is clear that the right hand side is contained in the left hand side of (6.91).

We will describe for each $\widehat{\psi}$ in the left hand side $\widehat{\mathfrak{m}_{b^{k(\ell,i)},\infty} \mid Y_1^{\ell,i}}$ the self-homeomorphisms $\overset{\circ}{\varphi} \in \widehat{\mathfrak{m}_{b,\infty} \mid \text{cyc}^{\ell,i}}$ for which $\widehat{\psi} = (\overset{\circ}{\varphi})^{k(\ell,i)} \mid Y_1^{\ell,i}$. For each $j = 1, \dots, k(\ell,i) - 1$, we take any homeomorphism $\overset{\circ}{\varphi}_j$ from $Y_j^{\ell,i}$ onto $Y_{j+1}^{\ell,i}$. For $j = k(\ell,i)$ we take the homeomorphism $\overset{\circ}{\varphi}_{k(\ell,i)}$ from $Y_{k(\ell,i)}^{\ell,i}$ onto $Y_1^{\ell,i}$ for which

$$(6.92) \quad \overset{\circ}{\varphi}_{k(\ell,i)} \circ \dots \circ \overset{\circ}{\varphi}_1 = \widehat{\psi} \quad \text{on} \quad Y_1^{\ell,i}.$$

Consider the self-homeomorphism $\overset{\circ}{\varphi}$ of $\bigcup_{j=1}^{k(\ell,i)} Y_j^{\ell,i}$ which equals $\overset{\circ}{\varphi}_j$ on $Y_j^{\ell,i}$. Prove

that it is in the class $\widehat{\mathfrak{m}_{b,\infty} \mid \text{cyc}^{\ell,i}}$. The restriction $(\overset{\circ}{\varphi})^{k(\ell,i)} \mid Y_1^{\ell,i}$ is in the class on the left hand side of equation (6.91). Indeed, equation (6.92) means

$$(\overset{\circ}{\varphi})^{k(\ell,i)} \mid Y_1^{\ell,i} = \widehat{\psi}.$$

Moreover, $\overset{\circ}{\varphi}$ and $\left(w \mid \bigcup_{j=1}^{k(\ell,i)} S_j^{\ell,i} \setminus E_n \right) \circ \varphi_b \circ \left(w \mid \bigcup_{j=1}^{k(\ell,i)} S_j^{\ell,i} \setminus E_n \right)^{-1}$ move the $Y_j^{\ell,i}$ along the same cycle, and $(\overset{\circ}{\varphi})^{k(\ell,i)} \mid Y_1^{\ell,i}$ and $w \circ \varphi_b^{k(\ell,i)} \circ w^{-1} \mid Y_1^{\ell,i}$ are in the same isotopy class. Hence, by the Corollary of the Lemma on conjugation the mapping $\overset{\circ}{\varphi}$ is in the conjugacy class of $\left(w \mid \bigcup_{j=1}^{k(\ell,i)} S_j^{\ell,i} \setminus E_n \right) \circ \varphi_b \circ \left(w \mid \bigcup_{j=1}^{k(\ell,i)} S_j^{\ell,i} \setminus E_n \right)^{-1}$, i.e. in $\widehat{\mathfrak{m}_{b,\infty} \mid \text{cyc}^{\ell,i}}$. It is clear that for each $\widehat{\psi}$ in the set on the left hand side of (6.91) all

elements $\overset{\circ}{\varphi}$ in the set on the right hand side of (6.91) for which $(\overset{\circ}{\varphi})^{k(\ell,i)} \mid Y_1^{\ell,i} = \overset{\circ}{\psi}$ are of the just described form. The lemma is proved. \square

Proof of the Lemma on conjugation. Put $\varphi_j = \varphi \mid Q_j$ and $\psi_j = \psi \mid Q_j$. We want to find self-homeomorphisms σ_j of Q_j such that the following diagram commutes:

$$\begin{array}{ccccccc} Q_1 & \xrightarrow{\psi_1} & Q_2 & \xrightarrow{\psi_2} & \dots & \xrightarrow{\psi_{k-1}} & Q_k & \xrightarrow{\psi_k} & Q_1 \\ \downarrow \sigma_1 & & \downarrow \sigma_2 & & & & \downarrow \sigma_k & & \downarrow \sigma_1 \\ Q_1 & \xrightarrow{\varphi_1} & Q_2 & \xrightarrow{\varphi_2} & \dots & \xrightarrow{\varphi_{k-1}} & Q_k & \xrightarrow{\varphi_k} & Q_1 \end{array}$$

Then σ will be the mapping whose restriction to Q_j equals σ_j , $j = 1, \dots, k$. The equality

$$(6.93) \quad \psi = \sigma^{-1} \circ \varphi \circ \sigma \quad \text{on} \quad \bigcup_{j=1}^k Q_j$$

is equivalent to the system of equations

$$(6.94) \quad \begin{aligned} \psi_1 &= \sigma_2^{-1} \circ \varphi_1 \circ \sigma_1 \\ &\vdots \\ \psi_{k-1} &= \sigma_k^{-1} \circ \varphi_{k-1} \circ \sigma_{k-1} \\ \psi_k &= \sigma_1^{-1} \circ \varphi_k \circ \sigma_k. \end{aligned}$$

Put $\sigma_1 = \text{id}$. Then the first equation in (6.94) is equivalent to

$$\sigma_2 = \varphi_1 \circ \psi_1^{-1}.$$

Successively, the j -th equation in (6.94) is equivalent to

$$(6.95) \quad \sigma_{j+1} = \varphi_j \circ \sigma_j \circ \psi_j^{-1} = \varphi_j \circ \dots \circ \varphi_1 \circ \psi_1^{-1} \circ \dots \circ \psi_j^{-1}, \quad j = 2, \dots, k-1.$$

With this equalities for σ_j , $j = 1, \dots, k-1$, the right hand side of the k -th equation in (6.94) equals

$$(6.96) \quad \begin{aligned} \sigma_1^{-1} \circ \varphi_k \circ \sigma_k &= \varphi_k \circ \sigma_k = \varphi_k \circ \varphi_{k-1} \circ \dots \circ \varphi_1 \circ \psi_1^{-1} \circ \dots \circ \psi_{k-1}^{-1} \\ &= (\varphi_k \circ \varphi_{k-1} \circ \dots \circ \varphi_1 \circ \psi_1^{-1} \circ \dots \circ \psi_{k-1}^{-1} \circ \psi_k^{-1}) \circ \psi_k. \end{aligned}$$

The equation $\varphi^k \mid Q_1 = \psi^k \mid Q_1$ is equivalent to the fact that the expression in brackets in (6.96) is equal to the identity on Q_k . Hence the right hand side of (6.96) equals ψ_k and hence the k -th equation of (6.94) holds.

The last assertion of the Lemma is clear. \square

Proof of Corollary 6.1. Let χ_0^t , $t \in [0, 1]$, be an isotopy, $\chi_0^t \in \text{Hom}^+ \left(\overline{Q_0}; \bigcup_{j=1}^k \partial Q_j \right)$,

$\chi_0^0 = \chi_0$, $\chi_0^1 = \text{id}$. Also, let χ_1^t , $t \in [0, 1]$, be an isotopy, $\chi_1^t \in \text{Hom}^+(\overline{Q_1}; \partial Q_1)$, $\chi_1^0 = \chi_1$, $\chi_1^1 = \text{id}$. Put φ_t equal to $\chi_0^t \circ (\psi \mid \overline{Q_0})$ on $\overline{Q_0}$, equal to ψ on $\overline{Q_j}$, $j = 1, \dots, k-1$, and equal to $\chi_1^t \circ \psi$ on $\overline{Q_k}$. Then φ_t , $t \in [0, 1]$, is an isotopy of

self-homeomorphisms in $\text{Hom}^+ \left(Q; \bigcup_{j=1}^k \partial Q_j \right)$ with $\varphi_1 = \psi$. For φ_0 we have the following conditions

$$\begin{aligned} \varphi_0 \mid \overline{Q_0} &= \varphi \mid \overline{Q_0} \\ \varphi_0^k \mid \overline{Q_1} &= \varphi^k \mid \overline{Q_1}. \end{aligned}$$

Moreover, φ_0 and φ permute the Q_j along the same cycle. It follows directly from the Lemma on conjugation that φ_0 is conjugate to ψ by a self-homeomorphism of Q which fixes the Q_j , $j = 0, \dots, k$, setwise and the union $\bigcup_{j=1}^k \partial Q_j$ pointwise. \square

CHAPTER 7

Proof of Theorem 1 for the reducible case

We will use the “analysis” and “synthesis” for reducible braids and for reducible mapping classes and will derive the statement of Theorem 1 for reducible braids.

We start with the following lemma which is a consequence of the “analysis” and “synthesis” for reducible braids. We use the notation of chapter 6.

For unifying notation in the following we write $k(1, 1) \stackrel{\text{def}}{=} k(1)$, $B(1, 1) \stackrel{\text{def}}{=} B(1)$, and use capital letters also in the case of pure braids: $B(\ell, j) \stackrel{\text{def}}{=} b(\ell, j)$ for pure braids.

LEMMA 7.1.

$$(7.1) \quad \mathcal{M}(\hat{b}) \geq \min_{\ell, j} \left(\mathcal{M} \left(\widehat{B(\ell, j)} \right) \cdot k(\ell, j) \right) .$$

Recall that $k(\ell, j) = 1$ for all ℓ and j if the braid is pure.

Proof of Lemma 7.1. Let

$$(7.2) \quad M_0 \stackrel{\text{def}}{=} \min_{\ell, j} \left(\mathcal{M} \left(\widehat{B(\ell, j)} \right) \cdot k(\ell, j) \right)$$

be the right hand side of the inequality (7.1) in the lemma. Let r_0 be a positive number such that $r_0 < e^{2\pi M_0}$. Consider the annulus

$$A_0 = \left\{ z \in \mathbb{C} : \frac{1}{\sqrt{r_0}} < |z| < \sqrt{r_0} \right\} .$$

The conformal module of A_0 equals

$$m A_0 = \frac{1}{2\pi} \log r_0 < M_0 .$$

Note that $k(1, 1) = k(1) = 1$, hence $\mathcal{M}(B(1)) \geq M_0$. Hence $B(1) = B(1, 1)$ can be represented by a holomorphic mapping of an annulus, containing the closure $\overline{A_0}$, into symmetrized configuration space. We obtain a holomorphic mapping

$$f^1 : A_0 \rightarrow C_{n(1)}(\mathbb{C}) / \mathcal{S}_{n(1)} ,$$

which extends continuously to the closure $\overline{A_0}$. Associate to f^1 the closed complex curve

$$(7.3) \quad \Gamma(f^1) = \{(\zeta, f^1(\zeta)) : \zeta \in A_0\}$$

in $A_0 \times \mathbb{C}$. Let $\Gamma_{2,1}$ be the connected component of this complex curve which corresponds to the cycle $\text{cyc}^{2,1}$ (i.e. after a free isotopy the intersection of this component with $\{1\} \times \mathbb{C}$ is equal to $\{z_1^{2,1}, \dots, z_{k(2,1)}^{2,1}\}$ for the points chosen in chapter 6 (see (6.62)). Denote by $\widetilde{A_0}^{k(2,1)}$ the $k(2, 1)$ -fold covering of A_0 . Let $\widetilde{\Gamma_{2,1}}$

be a suitable lift of $\Gamma_{2,1}$ to $\widetilde{A}_0^{k(2,1)} \times \mathbb{C}$. $\widetilde{\Gamma}_{2,1}$ is the graph of a function over $\widetilde{A}_0^{k(2,1)}$. Denote this function by $f_{2,1}$.

Note that

$$(7.4) \quad m(\widetilde{A}_0^{k(2,1)}) = \frac{1}{k(2,1)} m(A_0) = \frac{1}{k(2,1)} \cdot \frac{1}{2\pi} \log r_0 < \frac{1}{k(2,1)} M_0.$$

Since $\mathcal{M}(B(2,1)) \geq \frac{1}{k(2,1)} M_0$, the braid $B(1,2)$ can be represented by a holomorphic mapping

$$f^{2,1} : \widetilde{A}_0^{k(2,1)} \rightarrow C_{n(2,1)}(\mathbb{C}) / \mathcal{S}_{n(2,1)}$$

which extends continuously to the closure of $\widetilde{A}_{2,1}^{k(2,1)}$. Consider the covering map

$$(7.5) \quad p_{k(2,1)} : \widetilde{A}_0^{k(2,1)} \times \mathbb{C} \rightarrow A_0 \times \mathbb{C}.$$

There is a small positive number $\tilde{\varepsilon}^{2,1}$ such that the $\tilde{\varepsilon}^{2,1}$ -neighbourhood of $\Gamma_{2,1}$ does not intersect the other connected components of $\Gamma(f^1)$. Let $\varepsilon^{2,1} \in (0, \tilde{\varepsilon}^{2,1})$ be a small number. Denote by

$$(7.6) \quad f_{2,1} \boxplus \varepsilon^{2,1} f^{2,1} : \widetilde{A}_0^{k(2,1)} \rightarrow C_{n(2,1)}(\mathbb{C}) / \mathcal{S}_{n(2,1)}$$

the mapping whose value at $\zeta \in \widetilde{A}_0^{k(2,1)}$ is obtained by adding $f_{2,1}(\zeta)$ to each point of the unordered tuple $\varepsilon^{2,1} f^{2,1}(\zeta)$. Consider the complex curve $\Gamma(f_{2,1} \boxplus \varepsilon^{2,1} f^{2,1})$ in $\widetilde{A}^{k(2,1)} \times \mathbb{C}$ determined by (7.6). Take the projection

$$(7.7) \quad \Gamma(f_{2,1} \boxplus \varepsilon^{2,1} f^{2,1}) \stackrel{\text{def}}{=} p_{k(2,1)}(\Gamma(\widetilde{f_{2,1} \boxplus \varepsilon^{2,1} f^{2,1}}))$$

to $A_0 \times \mathbb{C}$ of the complex curve $\Gamma(f_{2,1} \boxplus \varepsilon^{2,1} f^{2,1})$. If $\varepsilon^{2,1}$ is small enough the set $\Gamma(f_{2,1} \boxplus \varepsilon^{2,1} f^{2,1})$ is contained in the $\frac{1}{2} \tilde{\varepsilon}^{2,1}$ -neighbourhood of $\Gamma_{2,1}$. Replace the connected component $\Gamma_{2,1}$ of $\Gamma(f^1)$ by $\Gamma(f_{2,1} \boxplus \varepsilon^{2,1} f^{2,1})$. We obtain a new closed complex curve in $A_0 \times \mathbb{C}$. It defines a holomorphic mapping from A_0 into symmetrized configuration space which we denote by

$$(7.8) \quad f^1 \sqcup (f_{1,2} \boxplus \varepsilon^{2,1} f^{2,1}) : A_0 \rightarrow C_{n'}(\mathbb{C}) / \mathcal{S}_{n'},$$

where $n' = n(1) + n(2,1) \cdot (k(2,1) - 1)$. The mapping extends continuously to $\overline{A_0}$.

By induction we construct a holomorphic mapping

$$(7.9) \quad f^1 \sqcup (f_{1,2} \boxplus \varepsilon^{2,1} f^{2,1}) \sqcup \dots \sqcup (f_{N,k_N} \boxplus \varepsilon^{N,k_N} f^{N,k_N})$$

which maps A_0 into $C_n(\mathbb{C}) / \mathcal{S}_n$, and represents

$$(7.10) \quad \hat{b} = \widehat{B(1)} \sqcup \widehat{B(2,1)} \sqcup \dots \sqcup \widehat{B(N,k_N)}.$$

Since the conformal module $m A_0 = \frac{1}{2\pi} \log r_0$ can be any number less than M_0 , we see that

$$(7.11) \quad \mathcal{M}(\hat{b}) \geq M_0.$$

□

We will now give a formula for the entropy of reducible mapping classes corresponding to braids in terms of the entropy of the irreducible nodal components. Let $b \in \mathcal{B}_n$ be a braid and $\mathfrak{m}_b \in \mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n)$ its mapping class. Let $\mathring{\mathfrak{m}}_b$ be the induced mapping class on an associated nodal surface Y . For computing entropy we will consider the closure $\overline{Y_1^{\ell,i}}$ in Y of each part of Y . For a cycle of parts we denote by

$\widehat{\text{cyc}}^{\circ, \ell, i}$ the cycle of closures of the parts. Recall that we identify self-homeomorphisms of Y with self-homeomorphisms of the closure \overline{Y} with distinguished points.

The following lemma is an easy consequence of Theorem 4 of [1].

LEMMA 7.2.

$$h\left(\widehat{\text{m}_{b, \infty}^{\circ}}\right) = \max_{\widehat{\text{cyc}}^{\circ, \ell, i}} h\left(\widehat{\text{m}_{b, \infty}^{\circ} \mid \widehat{\text{cyc}}^{\circ, \ell, i}}\right).$$

The following lemma allows to apply Theorem 3.2 to compute the entropy of the irreducible nodal components.

LEMMA 7.3.

$$h\left(\widehat{\text{m}_{b, \infty}^{\circ} \mid \widehat{\text{cyc}}^{\circ, \ell, i}}\right) = \frac{1}{k(\ell, i)} h\left(\widehat{\text{m}_{b^{k(\ell, i)}, \infty}^{\circ} \mid \overline{Y_1^{\ell, i}}}\right).$$

The following Theorem 7.1 is the main statement concerning the entropy of reducible braids. It expresses the entropy of a conjugacy class of a braid (rather than the entropy of its nodal mapping class) in terms of the entropy of the irreducible nodal components. Theorem 7.1 is a reformulation of Theorem 5 of the introduction. (Lemmas 7.2 and 7.3 imply that Theorem 7.1 and Theorem 5 are equivalent.)

THEOREM 7.1.

$$h(\widehat{\text{m}_b}) = \max_{\widehat{\text{cyc}}^{\circ, \ell, i}} \left(h\left(\widehat{\text{m}_{b^{k(\ell, i)}, \infty}^{\circ} \mid \overline{Y_1^{\ell, i}}}\right) \cdot \frac{1}{k(\ell, i)} \right).$$

Proof of Lemma 7.3. By definition $h\left(\widehat{\text{m}_{b, \infty}^{\circ} \mid \widehat{\text{cyc}}^{\circ, \ell, i}}\right)$ equals the infimum of entropies $h(\widehat{\varphi})$ where $\widehat{\varphi}$ ranges over all self-homeomorphisms of $\bigcup_{j=1}^{k(\ell, i)} \overline{Y_j^{\ell, i}}$ with set of distinguished points $(w(E_n) \cup \mathcal{N}) \cap \left(\bigcup_{j=1}^{k(\ell, i)} \overline{Y_j^{\ell, i}}\right)$, which are obtained from $(w \mid \text{cyc}^{\ell, i}) \circ \varphi_b \circ (w \mid \text{cyc}^{\ell, i})^{-1}$ by extension across the punctures, isotopy and conjugation. For such $\widehat{\varphi}$ we have

$$(7.12) \quad h(\widehat{\varphi}) = \frac{1}{k(\ell, i)} h((\widehat{\varphi})^{k(\ell, i)}).$$

Since $(\widehat{\varphi})^{k(\ell, i)} \mid Y_j^{\ell, i}$ is conjugate to $(\widehat{\varphi})^{k(\ell, i)} \mid Y_1^{\ell, i}$, $j = 1, \dots, k(\ell, i)$, we have

$$(7.13) \quad h(\widehat{\varphi}) = \frac{1}{k(\ell, i)} h\left((\widehat{\varphi})^{k(\ell, i)} \mid \overline{Y_1^{\ell, i}}\right).$$

The infimum over $\widehat{\varphi}$ is by Lemma 6.7 equal to

$$\frac{1}{k(\ell, i)} h\left(\widehat{\text{m}_{b^{k(\ell, i)}, \infty}^{\circ} \mid \overline{Y_1^{\ell, i}}}\right).$$

□

The proof of Theorem 7.1 will be split into two parts. The first part is the following.

LEMMA 7.4.

$$h(\widehat{\mathfrak{m}_b}) \leq \max_{\substack{\circ \\ \text{cyc} \\ \ell, i}} \left(h \left(\widehat{\mathfrak{m}_{b^{k(\ell, i)}, \infty} \mid \overline{Y_1^{\ell, i}}} \right) \cdot \frac{1}{k(\ell, i)} \right).$$

Proof. The proof is based on Theorem 4. For transparency we provide the proof first for pure braids. For the case of pure braids, all cycles have length $k(\ell, i) = 1$, and on the right hand side of the inequality of Lemma 7.4 we consider conjugacy classes of restrictions $\mathfrak{m}_{b, \infty} \mid \overline{Y_1^{\ell, i}}$ to nodal components. These restrictions are mapping classes in $\mathfrak{M}(\overline{Y^{\ell, i}}; w(E^{\ell, j}) \cup \mathcal{N}^{\ell, j})$, see (6.30) and (6.31). By (6.32), (6.33) and (6.35) the class $\mathfrak{m}_{b, \infty} \mid \overline{Y_1^{\ell, i}}$ is conjugate to

$$(7.14) \quad \mathfrak{m}_{b(\ell, j), \infty} = \mathcal{H}_{\zeta^{\ell, j}} \left(\mathfrak{m} \left(\varphi_b \mid \overline{S^{\ell, j}} \right) \right) \in \mathfrak{M}(\mathbb{P}^1; \zeta^{\ell, j}).$$

(φ_b is a homeomorphism in \mathfrak{m}_b which is completely reduced by the system \mathcal{C} .) Here $\zeta^{\ell, j} = \mathcal{E}^{\ell, j} \cup \{\infty\}$ and the subsets $S^{\ell, j}$, $E^{\ell, j}$ and $\mathcal{E}^{\ell, j}$ of \mathbb{C} have previous meaning. Consider an absolutely extremal mapping $\tilde{\varphi}^{\ell, j} \in \widehat{\mathfrak{m}_{b(\ell, j), \infty}}$. Thus

$$(7.15) \quad \begin{aligned} h(\tilde{\varphi}^{\ell, j}) &= \inf \{ h(\psi) : \psi \in \widehat{\mathfrak{m}_{b(\ell, j), \infty}} \} \\ &= \inf \left\{ h(\psi) : \psi \in \widehat{\mathfrak{m}_{b, \infty} \mid \overline{Y^{\ell, i}}} \right\}. \end{aligned}$$

The mapping $\tilde{\varphi}^{\ell, j}$ represents an element of the class $\mathfrak{M}(\mathbb{P}^1; \tilde{E}^{\ell, j} \cup \tilde{\zeta}^{\ell, j})$ obtained from an element of $\mathfrak{M}(\mathbb{P}^1; E^{\ell, j} \cup \zeta^{\ell, j})$ by conjugation with a self-homeomorphism $w^{\ell, j}$ of \mathbb{P}^1 , such that $w^{\ell, j}(\infty) = \infty$, $w^{\ell, j}(\mathcal{E}^{\ell, j}) = \tilde{\mathcal{E}}^{\ell, j}$, $w^{\ell, j}(\tilde{E}^{\ell, j}) = E^{\ell, j}$. Write $\tilde{\zeta}^{\ell, j} = \{\infty\} \cup \tilde{\mathcal{E}}^{\ell, j} = w^{\ell, j}(\zeta^{\ell, j})$. Apply Theorem 3.4 with $E' = \tilde{\zeta}^{\ell, j}$. We obtain a mapping $\tilde{\varphi}_0^{\ell, j}$ of the same entropy $h(\tilde{\varphi}_0^{\ell, j}) = h(\tilde{\varphi}^{\ell, j})$ which is isotopic to $\tilde{\varphi}^{\ell, j}$ in $\mathfrak{M}(\mathbb{P}^1; \tilde{\zeta} \cup \tilde{E}^{\ell, j})$ and equals the identity in a neighbourhood of $\{\infty\} \cup \tilde{\mathcal{E}}^{\ell, j}$.

To obtain a self-homeomorphism of \mathbb{P}^1 in the class $\widehat{\mathfrak{m}_{b(\ell, j), \infty}}$ which equals the identity in a neighbourhood of $\mathbb{P}^1 \setminus S^{\ell, j}$, we conjugate $\tilde{\varphi}_0^{\ell, j}$ by $(w^{\ell, j})^{-1} \circ (w_{\ell, j})$. The mapping $w_{\ell, j}$ is a self-homeomorphism of \mathbb{P}^1 in $\text{Hom}^+(\mathbb{P}^1; E^{\ell, j} \cup \zeta)$ which is isotopic to the identity. We choose $w_{\ell, j}$ so that the mapping $\varphi_0^{\ell, j}$ obtained by conjugation is equal to the identity in a neighbourhood of $\mathbb{P}^1 \setminus S^{\ell, j}$. We have

$$(7.16) \quad h(\varphi_0^{\ell, j}) = h(\tilde{\varphi}^{\ell, j}) = h \left(\widehat{\mathfrak{m}_{b, \infty} \mid \overline{Y^{\ell, i}}} \right),$$

and

$$(7.17) \quad \varphi_0^{\ell, j} \mid S^{\ell, j} \text{ is isotopic to } \varphi_b \mid S^{\ell, j} \text{ through homeomorphisms in } \text{Hom}^+(S^{\ell, j}; E^{\ell, j} \cup \zeta).$$

Hence the mapping classes in $\mathfrak{M}(\overline{S^{\ell, j}}; \partial S^{\ell, j} \cup E^{\ell, j})$ of these two maps differ by a product of powers of Dehn twists about simple closed curves that are homologous in $S^{\ell, j} \setminus E$ to boundary curves of $S^{\ell, j}$. Correct $\varphi_0^{\ell, j}$, using Lemma 3.7, to a map denoted again by $\varphi_0^{\ell, j}$, such that the mapping classes of $\varphi_0^{\ell, j} \mid \overline{S^{\ell, j}}$ and $\varphi_b \mid \overline{S^{\ell, j}}$ in $\mathfrak{M}(\overline{S^{\ell, j}}; \partial S^{\ell, j} \cup E^{\ell, j})$ coincide.

Denote by $\varphi_0 \in \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D}, E)$ the homeomorphism which equals $\varphi_0^{\ell,j} | \overline{S^{\ell,j}}$ for each (ℓ, j) . (Write $S^{1,1}$ for S^1 .) Then φ_0 has entropy

$$(7.18) \quad h(\varphi_0) = \max_{\ell,j} h\left(\widehat{\mathfrak{m}_b | Y^{\ell,j}}\right),$$

and φ_0 represents $\mathfrak{m}_b \in \mathfrak{M}(\overline{\mathbb{D}}; \partial\mathbb{D}, E)$. The lemma is proved for the case of pure braids.

We prove now the lemma in the general case. Consider the mapping classes $\mathfrak{m}_{b,\infty} | \overline{Y^{\ell,i}}$ by induction on the number of generation ℓ . Start with $\mathfrak{m}_{b,\infty} | \overline{Y^1}$. This is a mapping class in

$$(7.19) \quad \mathfrak{M}\left(\overline{Y^1}; w(z_\infty), w(E^1) \cup (\mathcal{N} \cap \overline{Y^1})\right),$$

see (6.43), (6.44) and (6.47). For a mapping $\varphi_b \in \mathfrak{m}_b$ which is completely reduced by the system of curves \mathcal{C} , the mapping class $\mathfrak{m}_{b,\infty} | \overline{Y^1}$ is conjugate to the class

$$(7.20) \quad \mathfrak{m}_{B(1),\infty} = \mathcal{H}_{\mathcal{E}^1 \cup \{\infty\}}\left(\mathfrak{m}\left(\varphi_b | \overline{S^1}\right)\right) \in \mathfrak{M}(\mathbb{P}^1; \infty, E^1 \cup \mathcal{E}^1),$$

see (6.48). Take an absolutely extremal mapping $\tilde{\varphi}^1$ in $\widehat{\mathfrak{m}_{B(1),\infty}}$. Then

$$(7.21) \quad h(\tilde{\varphi}^1) = h\left(\widehat{\mathfrak{m}_{b,\infty} | \overline{Y^1}}\right).$$

$\tilde{\varphi}^1$ represents a mapping class in $\mathfrak{M}(\mathbb{P}^1; \infty, \tilde{E}^1 \cup \tilde{\mathcal{E}}^1)$ for respective sets $\tilde{E}^1, \tilde{\mathcal{E}}^1$. Denote the conjugating homeomorphism by w^1 . So, $w^1(\infty) = \infty, w^1(\tilde{E}^1) = E^1, w^1(\tilde{\mathcal{E}}^1) = \mathcal{E}^1$.

Suppose the modular transformation of $\tilde{\varphi}^1$ is hyperbolic. Apply Theorem 3.4 to $\tilde{\varphi}^1$ with $E' = \{\infty\} \cup \tilde{\mathcal{E}}^1$. We obtain a mapping $\tilde{\varphi}_0^1$ which is isotopic to $\tilde{\varphi}^1$ through homeomorphisms in $\text{Hom}^+(\mathbb{P}^1; \infty, \tilde{\mathcal{E}}^1 \cup \tilde{E}^1)$, has entropy $h(\tilde{\varphi}_0^1) = h(\tilde{\varphi}^1)$ equal to that of $\tilde{\varphi}^1$, and has the following property. It equals the identity on a disc δ_∞ around ∞ ; it permutes topological discs δ_z around points $z \in \tilde{\mathcal{E}}^1$ in cycles corresponding to the cycles $\text{cyc}_{2,1}^{\circ}$ of the nodal components of generation 2. For each such cycle the iterate $(\tilde{\varphi}_0^1)^{k(2,1)}$ is the identity on the discs of this cycle.

If the modular transformation of $\tilde{\varphi}^1$ is elliptic then $\tilde{\varphi}^1$ is conformal and a power of $\tilde{\varphi}^1$ is the identity. Hence also a power of the iterate $(\tilde{\varphi}_0^1)^{k(2,1)}$ is the identity. Since ∞ is fixed by $\tilde{\varphi}^1$ the iterate $(\tilde{\varphi}_0^1)^{k(2,1)}$ moves $\tilde{\mathcal{E}}^1 \cup \tilde{E}^1$ along a single cycle. Hence the homeomorphism $\tilde{\varphi}^1$ automatically has the properties listed for the homeomorphism $\tilde{\varphi}_0^1$ in the hyperbolic case.

Conjugate $\tilde{\varphi}_0^1$ by the homeomorphism $(w^1)^{-1} \circ w_1$, where w_1 is isotopic to the identity through homeomorphisms in $\text{Hom}^+(\mathbb{P}^1; \infty, E^1 \cup \mathcal{E}^1)$. Moreover, w_1 is chosen so that it maps $\overline{S^1}$ onto $\mathbb{P}^1 \setminus \bigcup_{z \in \tilde{\mathcal{E}}^1 \cup \{\infty\}} w^1(\delta_z)$. Denote the conjugated map

by φ_0^1 . It has the same entropy

$$(7.22) \quad h(\varphi_0^1) = h(\tilde{\varphi}_0^1) = h(\tilde{\varphi}^1)$$

as $\tilde{\varphi}^1$. Notice that φ_0^1 equals the identity on $\partial_{\mathcal{E}} S^1 = \partial\mathbb{D}$. Further, if there is a cycle of components of $\partial_{\mathcal{J}} S^1$ of length k , then we have $(\varphi_0^1)^k = \text{id}$ on each set of the cycle.

Recall that we have chosen a representative φ_b of \mathfrak{m}_b which is completely reduced by \mathcal{C} . Change φ_b in a small neighbourhood of $\partial_{\mathcal{J}} S^1$ by an isotopy through

homeomorphisms in $\text{Hom}(\overline{\mathbb{D}}; \partial\mathbb{D}, E)$ so that $\varphi_b = \varphi_0^1$ on ∂S^1 . We may assume that the mapping φ_b was chosen from the beginning to have this property. Since $\varphi_b \mid S^1$ and $\varphi_0^1 \mid S^1$ are free isotopic on S^1 with distinguished points $E \cap S^1$, the mapping

$$(7.23) \quad \chi^1 = \varphi_0^1 \circ (\varphi_b)^{-1} \mid \overline{S^1} \in \text{Hom}(\overline{S^1}; \partial S^1, E \cap S^1)$$

is isotopic to a product of powers of Dehn twists about curves which are homologous to the boundary components of S^1 in $S^1 \setminus E$. By Lemma 3.7 we may change φ_0^1 by a product of powers of Dehn twists about curves close to the connected components of $\partial_{\mathcal{J}} S$ without changing the entropy. Hence, we may assume from the beginning that

$$(7.24) \quad \varphi_0^1 \mid \overline{S^1} = \chi^1 \circ (\varphi_b \mid \overline{S^1}),$$

where

$\chi^1 \mid \overline{S^1}$ is isotopic to the identity through homeomorphisms in

$$(7.25) \quad \text{Hom}(\overline{S^1}; \partial S^1, E \cap S^1).$$

Moreover, we have

$$(7.26) \quad h(\varphi_0^1 \mid \overline{S^1}) = h\left(\widehat{\mathfrak{m}_{b,\infty} \mid \overline{Y^1}}\right).$$

Consider now a cycle $\text{cyc}^{\circ 2,1}$ of generation 2. Consider $\mathfrak{m}_{b^{k(2,1)},\infty}^{\circ} \mid \overline{Y_1^{2,1}}$. This is a mapping class in

$$(7.27) \quad \mathfrak{M}\left(\overline{Y_1^{2,1}}; \{z_{\infty}\}, w(E^{2,1}) \cup (\mathcal{N}^{2,1} \setminus \{z_{\infty}\})\right).$$

Here z_{∞} is the node in $\mathcal{N}^{2,1}$ corresponding to the exterior boundary $\partial_{\mathfrak{E}} S_1^{2,1}$. By Lemma 6.5 the class $\mathfrak{m}_{b^{k(2,1)},\infty}^{\circ} \mid \overline{Y_1^{2,1}}$ is conjugate to

$$(7.28) \quad \mathfrak{m}_{B(2,1),\infty} = \mathcal{H}_{\mathcal{E}^{2,1} \cup \{\infty\}}\left(\mathfrak{m}\left(\varphi_{b^{k(2,1)}} \mid S_1^{2,1}\right)\right) \in \mathfrak{M}(\mathbb{P}^1; \infty, E^{2,1} \cup \mathcal{E}^{2,1}).$$

Here $\varphi_{b^{k(2,1)}} \in \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D}, E)$ is reduced by the system \mathcal{C} , represents $b^{k(2,1)}$, and equals $(\varphi_b)^{k(2,1)}$ on $\overline{S^1}$. Take an absolutely extremal mapping $\tilde{\varphi}^{2,1}$ in $\widehat{\mathfrak{m}_{B(2,1),\infty}^{\circ}}$. Thus

$$(7.29) \quad h(\tilde{\varphi}^{2,1}) = h\left(\widehat{\mathfrak{m}_{b^{k(2,1)},\infty}^{\circ} \mid \overline{Y^{2,1}}}\right).$$

$\tilde{\varphi}^{2,1}$ represents a mapping class in $\mathfrak{M}(\mathbb{P}^1; \infty, \tilde{E}^{2,1} \cup \tilde{\mathcal{E}}^{2,1})$ for respective sets $\tilde{E}^{2,1}$, $\tilde{\mathcal{E}}^{2,1}$. The class $\mathfrak{M}(\mathbb{P}^1; \infty, \tilde{E}^{2,1} \cup \tilde{\mathcal{E}}^{2,1})$ is conjugate to $\mathfrak{M}(\mathbb{P}^1; \infty, E^{2,1} \cup \mathcal{E}^{2,1})$. Denote the conjugating homeomorphism by $w^{2,1}$. We have $w^{2,1}(\infty) = \infty$, $w^{2,1}(\tilde{E}^{2,1}) = E^{2,1}$, $w^{2,1}(\tilde{\mathcal{E}}^{2,1}) = \mathcal{E}^{2,1}$. Suppose $\tilde{\varphi}^{2,1}$ has hyperbolic modular transformation. Theorem 3.4, applied to $\tilde{\varphi}^{2,1}$ and $E' = \{\infty\} \cup \tilde{\mathcal{E}}^{2,1}$, gives a homeomorphism $\tilde{\varphi}_0^{2,1}$ which is isotopic to $\tilde{\varphi}^{2,1}$ through homeomorphisms in $\text{Hom}^+(\mathbb{P}^1; \infty, \tilde{E}^{2,1} \cup \tilde{\mathcal{E}}^{2,1})$, has entropy $h(\tilde{\varphi}_0^{2,1}) = h(\tilde{\varphi}^{2,1})$, and has the following property. The map $\tilde{\varphi}^{2,1}$ equals the identity on a disc δ_{∞} around ∞ ; it permutes topological discs δ_z around $z \in \tilde{\mathcal{E}}^{2,1}$ in cycles corresponding to the cycles $\text{cyc}^{3,i}$ adjacent to $S_1^{2,1}$; for each such cycle the iterate $(\tilde{\varphi}_0^{2,1})^{k(3,i)}$ is the identity on the discs of this cycle.

Conjugate $\tilde{\varphi}_0^{2,1}$ by a homeomorphism $(w^{2,1})^{-1} \circ w_{2,1}$, where $w_{2,1}$ is chosen to have the following properties. $w_{2,1}$ is isotopic to the identity through homeomorphisms in $\text{Hom}^+(\mathbb{P}^1; \infty, E^{2,1} \cup \mathcal{E}^{2,1})$ and maps $\overline{S_1^{2,1}}$ onto $\mathbb{C} \setminus \bigcup_{z \in \mathcal{E}^{2,1} \cup \{\infty\}} w^{2,1}(\delta_z)$.

Denote the conjugated map by $\varphi_0^{2,1}$. Then

$$(7.30) \quad h(\varphi_0^{2,1}) = h\left(\tilde{\varphi}_0^{2,1}\right) = h(\tilde{\varphi}^{2,1}).$$

Also, $\varphi_0^{2,1} = \text{id}$ on $\partial_{\mathfrak{E}} S_1^{2,1}$. If there is a $(\varphi_b)^{k(2,1)}$ -cycle of components of $\partial_{\mathfrak{J}} S^{2,1}$ of length k , then we have $(\varphi_0^{2,1})^k = \text{id}$ on each set of this cycle. Make an isotopy of the previously chosen homeomorphism φ_b which changes φ_b only in a small neighbourhood of $\partial_{\mathfrak{J}} S_{k(2,1)}^{2,1}$ so that $\varphi_b^{k(2,1)}$ equals $\varphi_0^{2,1}$ on $\partial S_1^{2,1}$. After correcting $\varphi_0^{2,1}$ by a product of Dehn twists without increasing entropy, we have

$$(7.31) \quad \varphi_0^{2,1} \mid \overline{S_1^{2,1}} = \chi^{2,1} \circ \left(\varphi_b^{k(2,1)} \mid \overline{S_1^{2,1}} \right)$$

for a self-homeomorphism $\chi^{2,1}$ of $\overline{S_1^{2,1}}$ such that

$\chi^{2,1}$ is isotopic to the identity through homeomorphisms in

$$(7.32) \quad \text{Hom}\left(\overline{S_1^{2,1}}; \partial S_1^{2,1}, E \cap S_1^{2,1}\right).$$

Moreover, (see (7.29))

$$(7.33) \quad h(\varphi_0^{2,1}) = h\left(\widehat{\mathfrak{m}_{b^{k(2,1),\infty}} \mid \overline{Y_1^{2,1}}}\right).$$

Associate to $\varphi_0^{2,1}$ a self-homeomorphism $\varphi_{\text{cyc}}^{2,1}$ of $\overline{S_1^{2,1}} \cup \dots \cup \overline{S_{k(2,1)}^{2,1}}$ such that

$$(7.34) \quad (\varphi_{\text{cyc}}^{2,1})^{k(2,1)} \mid S_1^{2,1} = \varphi_0^{2,1}.$$

We proceed as follows. Define for each $j = 1, \dots, k(2,1) - 1$, a homeomorphism $\varphi_j^{2,1}$ from $\overline{S_j^{2,1}}$ onto $\overline{S_{j+1}^{2,1}}$ which maps the distinguished points $S_j^{2,1} \cap E$ onto the distinguished points $S_{j+1}^{2,1} \cap E$ and has the following additional property. The values of $\varphi_j^{2,1}$ on the boundary $\partial S_j^{2,1}$ coincide with those of φ_b . For the exterior boundary component $\partial_{\mathfrak{E}} S_j^{2,1}$ this means that the values of $\varphi_j^{2,1}$ coincide with those of φ_0^1 . For $j = k(2,1)$ we take the homeomorphism $\varphi_{k(2,1)}^{2,1} : \overline{S_{k(2,1)}^{2,1}} \rightarrow \overline{S_1^{2,1}}$ for which

$$(7.35) \quad \varphi_{k(2,1)}^{2,1} \circ \dots \circ \varphi_1^{2,1} = (\varphi_0^{2,1})^{k(2,1)} \quad \text{on} \quad S_1^{2,1}.$$

Automatically $\varphi_{k(2,1)}^{2,1}$ maps the distinguished points $\varphi_{k(2,1)}^{2,1} S_{k(2,1)}^{2,1} \cap E$ to the distinguished points $S_1^{2,1} \cap E$ because $\varphi_0^{2,1}$ fixes $S_1^{2,1} \cap E$ setwise. Moreover, $\varphi_{k(2,1)}^{2,1}$ coincides with φ_b on $\partial S_{k(2,1)}^{2,1}$, since $\varphi_0^{2,1} = \varphi_b^{k(2,1)}$ on $\partial S_1^{2,1}$. Thus, $\varphi_{k(2,1)}^{2,1}$ coincides with φ_0^1 on $\partial_{\mathfrak{E}} S_{k(2,1)}^{2,1}$. Define $\varphi_{\text{cyc}}^{2,1}$ to be equal to $\varphi_j^{2,1}$ on $\overline{S_j^{2,1}}$, $j = 1, \dots, k(2,1)$. Since

$$(7.36) \quad h(\varphi_{\text{cyc}}^{2,1}) = \frac{1}{k(2,1)} h\left((\varphi_{\text{cyc}}^{2,1})^{k(2,1)}\right),$$

and $(\varphi_{\text{cyc}}^{2,1})^{k(2,1)} \mid S_j^{2,1}$ is conjugate to $(\varphi_{\text{cyc}}^{2,1})^{k(2,1)} \mid S_1^{2,1} = \varphi_0^{2,1}$ for $j = 2, \dots, k(2, 1)$, we get from (7.33) and (7.34)

$$(7.37) \quad h(\varphi_{\text{cyc}}^{2,1}) = \frac{1}{k(2, 1)} h \left(\widehat{\mathfrak{m}_{b^{k(2,1)}} \mid \overline{Y^{2,1}}} \right).$$

Put $Q = (\overline{S^1} \setminus E) \cup \bigcup_{j=1}^{k(2,1)} (\overline{S_j^{2,1}} \setminus E)$, and $Q_j = (S_j^{2,1} \setminus E) \cup \partial_{\mathfrak{J}} S_j^{2,1}$, $j = 1, \dots, k(2, 1)$.

Also, let $Q_0 = Q \setminus \bigcup_{j=1}^{k(2,1)} Q_j$. Then Q is a topological space and Q_j , $j = 1, \dots, k(2, 1)$, are disjoint relatively open subsets. By $\overline{Q_j}$, $j = 0, \dots, k(2, 1)$, we mean the closure of Q_j in Q , i.e. for $j \geq 1$ $\overline{Q_j} = \overline{S_j^{2,1}} \setminus E$, and $\overline{Q_0} = \overline{S^1} \setminus E$. By ∂Q_j we mean the boundary in Q . I.e. for $j = 1, \dots, k(2, 1)$, $\partial Q_j = \partial_{\mathfrak{E}} S_j^{2,1}$, and $\partial Q_0 = \bigcup_{j=1}^{k(2,1)} \partial Q_j = \partial_{\mathfrak{J}} S^1$.

Define a self-homeomorphism φ_Q of Q as follows. Put $\varphi_Q \mid \overline{Q_0} = \varphi_0^1 \mid \overline{Q_0}$. For $j = 1, \dots, k(2, 1)$, we put $\varphi_Q \mid \overline{Q_j} = \varphi_j^{2,1} \mid \overline{Q_j}$.

The homeomorphisms $\varphi = \varphi_Q$ and $\psi = \varphi_b \mid Q$ satisfy the conditions of Corollary 6.1 by (7.24), (7.25), (7.31) and (7.32). Hence φ_Q is obtained from $\varphi_b \mid Q$ by isotopy through mappings in $\text{Hom}^+ \left(Q; \bigcup_{j=1}^{k(2,1)} \partial Q_j \right)$ followed by conjugation with a

self-homeomorphism of Q which fixes each Q_j setwise and fixes the union $\bigcup_{j=1}^{k(2,1)} \partial Q_j$ pointwise. By Theorem 4 of [1] and equations (7.26) and (7.37), we get

$$(7.38) \quad h(\varphi_Q) = \max \left\{ h \left(\widehat{\mathfrak{m}_{b,\infty} \mid \overline{Y^1}} \right), h \left(\widehat{\mathfrak{m}_{b^{k(2,1)},\infty} \mid \overline{Y_1^{2,1}}} \right) \cdot \frac{1}{k(2, 1)} \right\}.$$

By induction, following along the same lines, we find a self-homeomorphism φ_0 in $\text{Hom}^+(\mathbb{D}; \partial \mathbb{D}, E)$ which is obtained from a representative $\varphi_b \in \mathfrak{m}_b$ by conjugation with homeomorphisms in this space and isotopy through such homeomorphisms, such that

$$(7.39) \quad h(\varphi_0) = \max_{(\ell, i)} \frac{1}{k(\ell, i)} h \left(\widehat{\mathfrak{m}_{b^{k(\ell, i)},\infty} \mid \overline{Y_1^{\ell, i}}} \right).$$

Taking the infimum over all homeomorphisms in $\widehat{\mathfrak{m}_b}$ we obtain the statement of the lemma. \square

The following lemma states the remaining inequality of Theorem 7.1.

LEMMA 7.5.

$$h(\widehat{\mathfrak{m}_b}) \geq \max_{\substack{\circ \\ \text{cyc} \\ \ell, i}} \left(h \left(\widehat{\mathfrak{m}_{b^{k(\ell, i)},\infty} \mid \overline{Y_1^{\ell, i}}} \right) \cdot \frac{1}{k(\ell, i)} \right).$$

Proof. The proof in the case of **pure** braids is simple. Indeed, \mathfrak{m}_b is the isotopy class of a representing mapping φ_b in $\mathfrak{M}(\mathbb{D}; \partial \mathbb{D}, E)$. For each label (ℓ, j) the class $\mathfrak{m}_{b,\infty} \mid \overline{Y^{\ell, j}}$ is conjugate to $\mathfrak{m}_{b(\ell, j),\infty} = \mathcal{H}_{\zeta^{\ell, j}} \left(\mathfrak{m} \left(\varphi_b \mid \overline{S^{\ell, j}} \right) \right)$ (see (7.14)) with

$\zeta^{\ell,j} = \mathcal{E}^{\ell,j} \cup \{\infty\}$ chosen in chapter 6. The choice of the $\mathcal{E}^{\ell,j}$ was made so that $\mathcal{E}^{\ell,j} \subset E$. By corollary 3.2 we have

$$h(\widehat{\mathfrak{m}_{b(\ell,j),\infty}}) = h(\widehat{\mathfrak{m}_{b(\ell,j)}}).$$

Here $\mathfrak{m}_{b(\ell,j)}$ is the isotopy class in $\mathfrak{M}(\overline{\mathbb{D}}; \partial\mathbb{D}, E^{\ell,j} \cup \mathcal{E}^{\ell,j})$ of the following mapping. Take a representative φ_b of \mathfrak{m}_b which fixes $\partial S^{\ell,j}$ pointwise and extend it to $\overline{\mathbb{D}}$ by putting it equal to the identity on the discs $\delta^{\ell+1,j'} \subset \mathbb{D}$ bounded by the components of $\partial_j S^{\ell,j}$ and equal to φ_b on the annulus bounded by $\partial\mathbb{D}$ and $\partial_{\mathfrak{E}} S^{\ell,j}$. The thus obtained mapping $\tilde{\varphi}_b$ is isotopic to φ_b in $\mathfrak{M}(\overline{\mathbb{D}}; \partial\mathbb{D}, E^{\ell,j} \cup \mathcal{E}^{\ell,j})$. Thus

$$(7.40) \quad h(\mathfrak{m}_b) = \inf\{h(\varphi) : \varphi \in \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D} \cup E) \text{ and} \\ \text{isotopic to } \varphi_b \text{ through such homeomorphisms}\},$$

and

$$(7.41) \quad h(\mathfrak{m}_{b(\ell,j)}) = \inf\{h(\varphi) : \varphi \in \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D} \cup E^{\ell,j} \cup \mathcal{E}^{\ell,j}) \text{ and} \\ \text{isotopic to } \varphi_b \text{ through such homeomorphisms}\}.$$

Since the space which appears in (7.40) is contained in the space which appears in (7.41), the infimum in (7.40) is not smaller than the infimum in (7.41):

$$(7.42) \quad h(\mathfrak{m}_b) \geq h(\mathfrak{m}_{b(\ell,j)}) \quad \text{for each } \ell, j.$$

Consider now the general case when the braid b is **not necessarily pure**. Let N be the smallest natural number for which b^N is a pure braid. Notice that

$$(7.43) \quad h(\mathfrak{m}_{b^N}) \leq N \cdot h(\mathfrak{m}_b).$$

Indeed, for a chosen homeomorphism $\varphi_b \in \mathfrak{m}_b$

$$(7.44) \quad h(\mathfrak{m}_{b^N}) = \inf\{h(\varphi) : \varphi \text{ is isotopic to } \varphi_b^N \text{ in } \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D}, E)\},$$

and

$$(7.45) \quad N \cdot h(\mathfrak{m}_b) = \inf\{N \cdot h(\psi) = h(\psi^N) : \psi \text{ is isotopic to } \varphi_b \text{ in } \text{Hom}^+(\overline{\mathbb{D}}; \partial\mathbb{D}, E)\}.$$

(See [1], Corollary.)

Since the infimum in (7.45) is taken over a smaller class than in (7.44), we obtain (7.43). The homeomorphism φ_b was chosen to be completely reduced by an admissible system of curves \mathcal{C} . φ_b^N is again reduced by \mathcal{C} but may be not completely reduced. Take a larger admissible system $\mathcal{C}_1 \supset \mathcal{C}$ and a homeomorphism $\varphi_{b^N} \in \mathfrak{m}_{b^N}$ which is isotopic to $(\varphi_b)^N$ and equal to $(\varphi_b)^N$ near \mathcal{C} , and is completely reduced by \mathcal{C}_1 . We obtain a new nodal surface X and a surjection v onto X .

Suppose $\overset{\circ}{\text{cyc}}^{\ell,i}$ is a nodal cycle on the previous nodal surface Y such that the mapping class $\overset{\circ}{\mathfrak{m}}_{b^{k(\ell,i)},\infty} | \overline{Y_1^{\ell,i}}$ is pseudo-Anosov, i.e. there exists an absolutely extremal mapping $\tilde{\varphi}^{\ell,i}$ in this class. Any power of $\tilde{\varphi}^{\ell,i}$ is again pseudo-Anosov. Hence the class $\overset{\circ}{\mathfrak{m}}_{b^N,\infty} | \overline{Y_1^{\ell,i}}$ is pseudo-Anosov, since N is divisible by $k(\ell,i)$, $N = k(\ell,i) \cdot m(\ell,i)$. Moreover,

$$(7.46) \quad h\left(\overset{\circ}{\mathfrak{m}}_{b^N,\infty} | \overline{Y_1^{\ell,i}}\right) = h\left((\tilde{\varphi}^{\ell,i})^{m(\ell,i)}\right) = m(\ell,i) h(\tilde{\varphi}^{\ell,i})$$

since $(\tilde{\varphi}^{\ell,i})^{m(\ell,i)}$ represents $\mathring{\mathbf{m}}_{b^N,\infty} \mid \overline{Y_1^{\ell,i}}$. We obtained

$$(7.47) \quad h\left(\widehat{\mathring{\mathbf{m}}_{b^N,\infty} \mid \overline{Y_1^{\ell,i}}}\right) = m(\ell,i) h\left(\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty} \mid \overline{Y_1^{\ell,i}}}\right).$$

Since X contains a nodal part corresponding to $Y_1^{\ell,i}$ we obtain, by the lemma applied to pure braids, and by (7.43) and (7.47)

$$(7.48) \quad N \cdot h(\mathbf{m}_b) \geq h(\widehat{\mathbf{m}_{b^N}}) \geq h\left(\widehat{\mathring{\mathbf{m}}_{b^N,\infty} \mid \overline{Y_1^{\ell,i}}}\right) = m(\ell,i) h\left(\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty} \mid \overline{Y_1^{\ell,i}}}\right).$$

This gives an inequality like the one stated in the lemma where the right hand side ranges over all nodal cycles for which the $\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty} \mid \overline{Y_1^{\ell,i}}$ is pseudo-Anosov. All other cycles of the nodal surface X correspond to the elliptic case, in which the entropy of the class is zero. Lemma 7.5 and, hence, Theorem 7.1 is proved. \square

We obtained so far

$$\mathcal{M}(\hat{b}) \geq \min_{\ell,j} \left(\mathcal{M}\left(\widehat{B(\ell,j)}\right) \right) \cdot k(\ell,j)$$

(see (7.1)),

$$h(\widehat{\mathbf{m}_b}) = h(\hat{b}) = \max_{\ell,j} \left(h\left(\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty} \mid \overline{Y_1^{\ell,i}}}\right) \cdot \frac{1}{k(\ell,i)} \right),$$

(see Theorem 7.1),

$$\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty} \mid \overline{Y_1^{\ell,i}}} = \widehat{\mathbf{m}_{B(\ell,i),\infty}},$$

(see Lemma 6.5). By Theorem 1 in the irreducible case we have

$$(7.49) \quad k(\ell,i) \mathcal{M}\left(\widehat{B(\ell,i)}\right) = \frac{\pi}{2} \frac{k(\ell,i)}{h\left(\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty} \mid \overline{Y_1^{\ell,i}}}\right)}.$$

We obtain the following inequality

$$\mathcal{M}(\hat{b}) \geq \frac{\pi}{2} \frac{1}{h(\widehat{\mathbf{m}_b})} = \frac{\pi}{2} \frac{1}{h(\hat{b})}.$$

It remains to prove the opposite inequality:

LEMMA 7.6.

$$\mathcal{M}(\hat{b}) \leq \frac{\pi}{2} \frac{1}{h(\hat{b})}.$$

Proof. By Proposition 4.1

$$\mathcal{M}(\hat{b}) \leq \frac{\pi}{2} \frac{1}{L(\varphi_{b,\infty}^*)}.$$

Here again $\varphi_{b,\infty}$ is a self-homeomorphism of \mathbb{P}^1 with the following properties. It is equal to the identity outside $\overline{\mathbb{D}}$ and on $\overline{\mathbb{D}}$ it is equal to a homeomorphism φ_b representing the mapping class \mathbf{m}_b . As before $\varphi_{b,\infty}^*$ denotes the modular transformation of $\varphi_{b,\infty}$. We have

$$L(\varphi_{b,\infty}^*) = \frac{1}{2} \log I(\varphi_{b,\infty})$$

(see (2.33) and (2.34)). As above, $I(\varphi_{b,\infty})$ is the infimum of quasiconformal dilations in the class obtained from $\varphi_{b,\infty}$ by isotopy and conjugation.

We may assume that $\varphi_{b,\infty}$ is completely reduced by a system of curves \mathcal{C} . Let Y be a nodal surface associated to the system \mathcal{C} . Let $\tilde{w} : Y \rightarrow \tilde{w}(Y) = \tilde{Y}$ be the conformal structure of part (1) of Theorem 2.11 and let $\tilde{\varphi}$ be the absolutely extremal self-homeomorphism of \tilde{Y} which appears in (1), Theorem 2.11. We have

$$\frac{1}{2} \log K(\tilde{\varphi}) = \frac{1}{2} \log I(\varphi_{b,\infty})$$

Then for the $\tilde{\varphi}$ -cycles $\text{cyc}^{\ell,i}$ of \tilde{Y} we have

$$(7.50) \quad \frac{1}{2} \log K(\tilde{\varphi}) = \max_{\ell,i} \left(\frac{1}{2} \log K(\tilde{\varphi} \mid \text{cyc}^{\ell,i}) \right).$$

By Lemma 2.5

$$(7.51) \quad \frac{1}{2} \log K(\tilde{\varphi} \mid \text{cyc}^{\ell,i}) = \frac{1}{2} \frac{1}{k(\ell,i)} \log K(\tilde{\varphi}^{k(\ell,i)} \mid \tilde{Y}_1^{\ell,i}) = \frac{1}{2} \frac{1}{k(\ell,i)} \log I(\tilde{\varphi}^{k(\ell,i)} \mid \tilde{Y}_1^{\ell,i}).$$

By the Theorem of Fathi-Shub for irreducible self-homeomorphisms of connected Riemann surfaces of first kind (see Theorem 3.2) we obtain

$$(7.52) \quad \frac{1}{2} \frac{1}{k(\ell,i)} \log K(\tilde{\varphi}^{k(\ell,i)} \mid \tilde{Y}_1^{\ell,i}) = \frac{1}{k(\ell,i)} h(\tilde{\varphi}^{k(\ell,i)} \mid \tilde{Y}_1^{\ell,i}).$$

Notice that

$$(7.53) \quad h(\tilde{\varphi}^{k(\ell,i)} \mid \tilde{Y}_1^{\ell,i}) = h\left(\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty}} \mid Y_1^{\ell,i}\right),$$

since $\tilde{\varphi}^{k(\ell,i)}$ is an extremal element of the class $\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty}$ induced by $\varphi_{b,\infty}$ on Y . By Lemma 7.2 and Lemma 7.3

$$(7.54) \quad \max_{\ell,i} \frac{1}{k(\ell,i)} h\left(\widehat{\mathring{\mathbf{m}}_{b^{k(\ell,i)},\infty}} \mid Y_1^{\ell,i}\right) = h\left(\widehat{\mathring{\mathbf{m}}_{b,\infty}}\right).$$

Hence, by (7.50), (7.51), (7.52), (7.53) and (7.54) we have

$$\frac{1}{2} \log I(\varphi_{b,\infty}) = \frac{1}{2} \log K(\tilde{\varphi}) = h(\widehat{\mathring{\mathbf{m}}_{b,\infty}}).$$

By Theorem 7.1 the entropy $h\left(\widehat{\mathring{\mathbf{m}}_{b,\infty}}\right)$ equals $h(\widehat{\mathbf{m}}_b) = h(\hat{b})$. Hence

$$(7.55) \quad \mathcal{M}(\hat{b}) \leq \frac{\pi}{2} \frac{1}{h(\widehat{\mathring{\mathbf{m}}_{b,\infty}})} = \frac{\pi}{2} \frac{1}{h(\widehat{\mathbf{m}}_b)} = \frac{\pi}{2} \frac{1}{h(\hat{b})}.$$

Lemma 7.6, and, hence, the general case of Theorem 1 is proved. \square

COROLLARY 7.1.

$$\mathcal{M}(\hat{b}) = \min_{\ell,j} \left(\mathcal{M}\left(\widehat{B(\ell,j)}\right) \cdot k(\ell,j) \right).$$

We give now the proof of Corollary 1 of the introduction.

Proof of Corollary 1. Corollary 1 is equivalent to the equality

$$h(\widehat{\mathfrak{m}_{b^\ell}}) = |\ell| h(\widehat{\mathfrak{m}_b})$$

for each braid and each non-zero integer. The equality is easy for irreducible braids. If the class $\widehat{\mathfrak{m}_b}$ is elliptic, the class $\widehat{\mathfrak{m}_{b^\ell}}$ is so and both sides are zero. If the class $\widehat{\mathfrak{m}_b}$ is pseudo-Anosov and $\tilde{\varphi}$ is a pseudo-Anosov representative then $\tilde{\varphi}^\ell$ is a pseudo-Anosov representative of $\widehat{\mathfrak{m}_{b^\ell}}$. The equality follows from

$$h(\tilde{\varphi}^\ell) = |\ell| h(\tilde{\varphi}).$$

(See [1], Corollary).

If b is reducible one has to apply this argument to each cycle of parts of a nodal surface associated to the class \mathfrak{m}_b and an admissible system of curves \mathcal{C} which completely reduces a representative of \mathfrak{m}_b . The details appeared also in the proof of Lemma 7.5 and are left to the reader. \square

CHAPTER 8

Applications to algebroid functions

The conformal module of braids and first applications of this concept to algebroid functions appeared in a line of research that was initiated by the 13th Hilbert problem, posed by Hilbert in his famous talk on the International Congress of Mathematicians 1900. Hilbert asked whether each holomorphic function in three variables can be written as finite composition of continuous functions of two variables. The question was answered affirmatively by Kolmogorov and Arnol'd. In the Proceedings of the Symposium [31] devoted to “Mathematical developments arising from Hilbert Problems”, the problem is commented as follows.

“Hilbert posed this question especially in connection with the solution of a general algebraic equation of degree 7. It is reasonable to presume that he formulated it in terms of continuous functions partly because he had an interest in nomography and partly because he expected a negative answer. Now that it is settled affirmatively, one can ask an equally fundamental, and perhaps more interesting, question with algebraic functions instead of continuous functions.”

This latter question goes back to mathematics of 17 century. Consider an algebraic equation of degree n .

$$(8.1) \quad z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0.$$

Put $a = (a_0, \dots, a_{n-1}) \in \mathbb{C}^n$ and consider

$$P(z, a) = z^n + a_{n-1} z^{n-1} + \dots + a_0$$

as polynomial of degree n whose coefficients are polynomials in a . In other words P is an algebraic function of degree n on \mathbb{C}^n . Parametrizing the space $\overline{\mathfrak{P}}_n$ by unordered n -tuples of zeros of polynomials, we get a map which assigns to each $a \in \mathbb{C}^n$ the n -tuple $\{z_1(a), \dots, z_n(a)\}$ of solutions of the equation $P(z, a) = 0$, i.e. what is called an “ n -valued algebraic function”.

Tschirnhaus suggested to “substitute” z by an algebraic function of a new variable w . In other words, put

$$(8.2) \quad w = z^k + b_{k-1} z^{k-1} + \dots + b_0,$$

and let $b = (b_0, \dots, b_{k-1}) \in \mathbb{C}^k$. To eliminate z from the two equations (8.1) and (8.2) consider P and Q , $Q(z, w) \stackrel{\text{def}}{=} z^k + b_{k-1} z^{k-1} + \dots + b_0 - w$, as polynomials in z with coefficients being polynomials in w . They are relatively prime. The resultant R is a polynomial in w (and in the coefficients a and b). It can be written as

$$R = pP + qQ$$

where p and q are polynomials in z and w (and in the coefficients $a \in \mathbb{C}^n$, $b \in \mathbb{C}^k$). If z satisfies (8.1) and (8.2) then $R(w) = 0$. This allows to think about solutions

of (8.1) as part of zeros of a “composition of algebraic functions” determined by the equations $Q(z, w) = 0$, respectively by $R(w) = 0$. The method consists now in choosing the coefficients b depending on a so that the “algebraic functions” depend on a smaller number of variables. This method brings the general equation of degree 7 to the equation

$$(8.3) \quad z^7 + a z^3 + b z^2 + c z + 1 = 0.$$

The 7-tuple of solutions of (8.3) is an algebraic function of degree 7 depending on three complex variables. Hilbert was interested in the “complexity” of the general 7-valued algebraic function, in other words, he wanted to know how far one can go in this process to get formulas beyond radicals.

There are two difficulties. First, the “function-composition” may have more zeros than P , and, secondly, the “algebraic functions” obtained by choosing the coefficients b are actually polynomials in one variable with coefficients being rational functions in several complex variables, that may have indeterminacy sets. The composition problem was rigorously formulated for entire algebraic functions (i.e. for polynomials in one variable with coefficients being entire functions of complex variables – the restricted composition problem) and was considered first for the case when the “function-composition” coincides with the original function (the faithful composition problem). Arnol’d’s interest in the topological invariants of the space \mathfrak{P}^n was motivated by finding cohomological obstructions for the restricted faithful composition problem. The restricted composition problem is now answered negatively in a series of papers by several authors. The last step was done by Lin (see [27]). But the original question allowing polynomials in one variable with coefficients being rational functions is widely open. Compare also with the more classical account [28].

In whatever sense one understands “algebraic functions”, for each algebraic function there is a Zariski open set in \mathbb{C}^n such that the restriction of the algebraic function to this set is a separable algebroid function. The restriction of the algebraic function to each loop in this set defines a conjugacy class of braids. Moreover, this conjugacy class is represented by a holomorphic mapping of an annulus into the space of polynomials – any annulus which is holomorphically embedded into the Zariski open set and represents the homotopy class of the loop may serve. We do not know at the moment whether further progress related to the concept of the conformal module of braids may have some impact on the open problems related to the 13th Hilbert problem. I am grateful to M. Zaidenberg who asked me this question.

We will focus in this chapter on applications of the concept of conformal module to algebroid functions, in particular to the question of existence of isotopies of separable quasipolynomials to algebroid functions.

We will prove here Theorem 2. This theorem will follow from several other theorems which are of independent interest and give more detailed information. Notice that the results for quasipolynomials apply to elliptic fiber bundles (see chapter 9).

In the end of the chapter we give a short conceptional proof and a slight improvement of a theorem of Gorin, Lin, Zjuzin ([15], [41]), and Petunin. This theorem is one of the first applications of the concept of conformal module to questions in algebraic geometry. The present proof of the theorem uses Theorem 1 and

a result of Penner [30] on the smallest non-vanishing entropy among irreducible n -braids.

We start with several lemmas and propositions which prepare the theorems.

LEMMA 8.1. *Suppose the separable quasipolynomial f of degree n on an annulus A is irreducible and n is prime. Then the induced conjugacy class of braids $\widehat{b}_{f,A}$ is irreducible.*

Notice that, on the other hand, conjugacy classes of irreducible pure braids define solvable, hence reducible, quasipolynomials on the circle.

Proof. If f is irreducible the conjugacy class of braids $\widehat{b}_{f,A}$ projects to a conjugacy class of n -cycles. The lemma is now a consequence of the following known lemma (see e.g. [11]).

LEMMA 8.2. *If n is prime then any braid $b \in \mathcal{B}_n$, for which $\tau_n(b)$ is an n -cycle, is irreducible.*

For convenience of the reader we give the short argument.

Proof of Lemma 8.2. If b was reducible then a homeomorphism φ which represents the mapping class corresponding to b would fix setwise an admissible system of curves \mathcal{C} . Let C_1 be one of the curves in \mathcal{C} and let δ_1 be the topological disc contained in \mathbb{D} , bounded by C_1 . δ_1 contains at least two distinguished points z_1 and z_2 . Since φ permutes the distinguished points along an n -cycle there is a power φ^k of φ which maps z_1 to z_2 . Since n is prime, φ^k also permutes the distinguished points along an n -cycle. Hence it maps some distinguished point in δ_1 to the complement of δ_1 . We obtained that $\varphi^k(\delta_1)$ intersects both, δ_1 and its complement. Hence $\varphi^k(C_1) \neq C_1$ but $\varphi^k(C_1)$ intersects C_1 . This contradicts the fact that the system of curves \mathcal{C} was admissible and invariant under φ . \square

It is known that the smallest non-vanishing entropy among irreducible 3-braids equals $\log \frac{3+\sqrt{5}}{2}$ (see e.g. [34]). Put $\eta = \frac{\pi}{2} \left(\log \frac{3+\sqrt{5}}{2} \right)^{-1}$. The following lemma holds.

LEMMA 8.3. *Suppose for a conjugacy class of braids $\widehat{b} \in \widehat{\mathcal{B}}_3$ the conformal module $\mathcal{M}(\widehat{b})$ satisfies the inequality $\mathcal{M}(\widehat{b}) > \eta$. Then the following holds.*

- (a) $\mathcal{M}(\widehat{b}) = \infty$.
- (b) *If \widehat{b} is irreducible then \widehat{b} is the conjugacy class of a periodic braid, i.e. either of $(\sigma_1 \sigma_2)^\ell$ for an integer ℓ , or of $(\sigma_1 \sigma_2 \sigma_1)^\ell$ for an integer ℓ .*
- (c) *If \widehat{b} is reducible then \widehat{b} is the conjugacy class of $\sigma_1^k \Delta_3^{2\ell}$ for integers k and ℓ . Here $\Delta_3^2 = (\sigma_1 \sigma_2 \sigma_1)^2 = (\sigma_1 \sigma_2)^3$.*
- (d) *If \widehat{b} is the conjugacy class of a commutator, then it is represented by a pure braid b .*

Proof. Suppose \widehat{b} is irreducible. Then by the definition of η we have $h(\widehat{b}) = 0$. By Theorem 1 we obtain $\mathcal{M}(\widehat{b}) = \infty$. This proves (a) in the irreducible case.

Let $b \in \mathcal{B}_3$ represent the irreducible class \widehat{b} , let \mathbf{m}_b be the mapping class of the 3-punctured disc corresponding to b and let $\mathcal{H}_\infty(\mathbf{m}_b)$ be the corresponding mapping class in the 3-punctured plane. This class is represented by a periodic mapping of the complex plane \mathbb{C} with three distinguished points. By a conjugation we may assume that the periodic mapping fixes zero and hence is a rotation by a root

of unity. If zero is not a distinguished point, the three distinguished points are equidistributed on a circle with center zero and the mapping is rotation by a power of $e^{\frac{2\pi i}{3}}$. This mapping corresponds to

$$(\widehat{\sigma_1 \sigma_2})^\ell / \langle \Delta_3^2 \rangle.$$

If zero is a distinguished point, the other two distinguished points are equidistributed on a circle with center zero. The mapping is a rotation, precisely, a multiplication by a power of -1 , and corresponds to $(\widehat{\sigma_1 \sigma_2 \sigma_1})^\ell / \langle \Delta_3^2 \rangle$. Note that if ℓ is even the braid b is reducible. We proved (b).

Suppose \hat{b} is reducible, i.e. the mapping class \mathfrak{m}_b is reducible for $b \in \hat{b}$. A reducing curve for a representing mapping $\varphi \in \mathfrak{m}_b$ is admissible, hence must separate two punctures from the third one and from the boundary $\partial \mathbb{D}$ of the disc. Multiply b by a power of Δ_3^2 so that the outer braid becomes trivial. The inner braid is a power of σ . In other words, for some integer ℓ the braid $b \Delta_3^{-2\ell}$ is conjugate to σ_1^k for an integer k . We obtained (c). The braid $b = \sigma_1^k \Delta_3^{2\ell}$ has infinite conformal module. This gives (a) in the reducible case.

Suppose \hat{b} is the conjugacy class of a commutator $b \in \mathcal{B}_3$. Then b has degree zero. If $\mathcal{M}(\hat{b}) = \infty$ then this is possible only if $b = \sigma^k \Delta_3^{2\ell}$ with $k + 6\ell = 0$. Hence, k is even and the braid b is pure. This proves (d). The lemma is proved. \square

PROPOSITION 8.1. *Let \hat{b} be a reducible conjugacy class of n -braids of infinite conformal module which is not periodic. Then each annulus of finite conformal module admits a holomorphic mapping into \mathfrak{P}_n , representing \hat{b} . But there is no holomorphic mapping from $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ into \mathfrak{P}_n representing \hat{b} . A holomorphic representing mapping from $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ exists in some cases.*

Proof. We have to prove that there is no holomorphic mapping from \mathbb{C}^* into \mathfrak{P}_n which represents \hat{b} . Assume the contrary. We may assume that \hat{b} is represented by a pure braid b . Indeed, if there is a holomorphic mapping from \mathbb{C}^* into \mathfrak{P}_n representing \hat{b} , then there is also such a holomorphic mapping representing \hat{b}^k for any $k \in \mathbb{Z} \setminus \{0\}$. The number k can be chosen so that b^k is pure.

Let $\mathfrak{m}_b \in \mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n)$ be the mapping class associated to the pure braid b . Let C be a simple closed Jordan curve, $C \subset \mathbb{D} \setminus E_n$, which reduces a homeomorphism φ_b in \mathfrak{m}_b , such that the restriction of φ_b to the relatively compact connected component of $\mathbb{D} \setminus C$ represents an irreducible mapping class. Since this irreducible mapping class fixes the distinguished points pointwise and has infinite conformal module it is conjugate to a power of a Dehn twist about the Jordan curve, hence, since the restriction to the connected component is irreducible, it is conjugate to the mapping class of $\Delta_2^{2\ell}$. We may assume that $n = 3$ and the relatively compact connected component of $\mathbb{D} \setminus C$ contains the two distinguished points corresponding to the first and second strand of b and the remaining connected component contains a single distinguished point. Indeed, if the pure braid class \hat{b} on n strand can be represented by a holomorphic mapping from \mathbb{C}^* into \mathfrak{P}_n then the braid class obtained by removing all but three strands can be represented by a holomorphic mapping from \mathbb{C}^* into \mathfrak{P}_3 .

Consider the holomorphic vector valued function

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} : \mathbb{C}^* \rightarrow C_3(\mathbb{C}) \subset \mathbb{C}^3$$

which represents the conjugacy class $\hat{b} \in \hat{\mathcal{B}}_3$ at which we arrived. There is a holomorphic family $\mathbf{a}(z) \in \mathcal{A}$, $z \in \mathbb{C}^*$, each $\mathbf{a}(z)$ being a complex affine self-homeomorphism of \mathbb{C} , such that $\mathbf{a}(z)(f_1(z)) \equiv 0$, $\mathbf{a}(z)(f_2(z)) \equiv 1$. Put $g(z) = \mathbf{a}(z)(f_3(z))$. g is a holomorphic function on \mathbb{C}^* which omits 0 and 1. The index of g with respect to 0 is equal to the index of g with respect to 1 (in view of the way the braid b is reduced) and this index is different from zero (because b is not periodic). Let this index be equal to j .

Consider the universal covering \mathbb{C}_+ of $\mathbb{C} \setminus \{0, 1\}$, and its covering $\mathbb{C}_+ / (\zeta \sim \zeta + 2)$,

$$\mathbb{C}_+ \rightarrow \mathbb{C}_+ / (\zeta \sim \zeta + 2j) \rightarrow \mathbb{C} \setminus \{0, 1\}.$$

Notice that $\mathbb{C}_+ / (\zeta \sim \zeta + 2j)$ is conformally equivalent to a punctured disc. Since the values of the index of g with respect to 0 and with respect to 1 equal j the mapping

$$g : \mathbb{C}^* \rightarrow \mathbb{C} \setminus \{0, 1\}$$

lifts to a mapping

$$\tilde{g} : \mathbb{C}^* \rightarrow \mathbb{C}_+ / (\zeta \sim \zeta + 2j) \cong \mathbb{D}^*$$

of index 1. Hence \tilde{g} extends to a holomorphic map from \mathbb{C} into \mathbb{D} . Hence, \tilde{g} must be constant, which is impossible. We proved that \hat{b} cannot be represented by a holomorphic map from \mathbb{C}^* into \mathfrak{P}_n .

Notice, that however, the considered 3-braid $b \in \mathcal{B}_3$ can be represented by a holomorphic mapping of the punctured disc into \mathfrak{P}_3 , i.e. of an annulus of conformal module being exactly equal to $\mathcal{M}(\hat{b})$. \square

For the proof of Theorem 2 in the irreducible case we need the following lemma on permutations. Its proof can be extracted for instance from [38]. For convenience of the reader we give the short argument.

LEMMA 8.4. *Let n be a prime number. Any Abelian subgroup of the symmetric group \mathcal{S}_n which acts transitively on a set consisting of n points is generated by an n -cycle.*

Proof. Let \mathcal{S}_n be the group of permutations of elements of the set $\{1, \dots, n\}$. Suppose the elements $s_j \in \mathcal{S}_n$, $j = 1, \dots, m$, commute and the subgroup $\langle s_1, \dots, s_m \rangle$ of \mathcal{S}_n generated by the s_j , $j = 1, \dots, m$, acts transitively on the set $\{1, \dots, n\}$. Let $A_{s_1} \subset \{1, \dots, n\}$ be a minimal s_1 -invariant subset. Then $s_1|_{A_{s_1}}$ is a cycle of length $k_1 = |A_{s_1}|$. (The order $|A|$ of a set A is the number of elements of this set.) For any integer ℓ the set $s_2^\ell(A_{s_1})$ is minimal s_1 -invariant. Hence, two such sets are either disjoint or equal. Take the minimal union $A_{s_1 s_2}$ of sets of the form $s_2^\ell(A_{s_1})$ for some integers ℓ which contains A_{s_1} and is invariant under s_2 . Then s_2 moves the $s_2^\ell(A_{s_1})$ along a cycle of length k_2 such that $|A_{s_1 s_2}| = k_1 \cdot k_2$. $A_{s_1 s_2}$ is a minimal subset of $\{1, \dots, n\}$ which is invariant for both, s_1 and s_2 . Continue in this way. We obtain a minimal set $A = A_{s_1 \dots s_m} \subset \{1, \dots, n\}$ which is invariant under all s_j . By the transitivity condition $A = \{1, \dots, n\}$. The order $|A|$ equals $k_1 \cdot \dots \cdot k_m$. Since n is prime, exactly one of the factors, k_{j_0} equals n , the other factors equal 1.

Then $|A_{s_1 \dots s_{j_0-1}}| = 1$, and s_{j_0} is a cycle of length n . Since each s_j commutes with s_{j_0} each s_j is a power of s_{j_0} . Indeed, consider a bijection of $\{1, \dots, n\}$ onto the set of n -th roots of unity so that s_{j_0} corresponds to rotation by the angle $\frac{2\pi}{n}$. In other words, put $\zeta = e^{\frac{2\pi i}{n}}$. The permutation s_{j_0} acts on $\{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}$ by multiplication by ζ . Consider an arbitrary s_j . Then for some integer ℓ_j , $s_j(\zeta) = \zeta^{\ell_j}$, i.e. $s_j(\zeta) = \zeta^{\ell_j-1} \cdot \zeta = (s_{j_0})^{\ell_j-1}(\zeta)$. Then for any other n -th root of unity ζ^m

$$s_j(\zeta^m) = s_j((s_{j_0})^{m-1}(\zeta)) = (s_{j_0})^{m-1}(s_j(\zeta)) = \zeta^{m-1} \cdot \zeta^{\ell_j} = \zeta^{\ell_j-1} \cdot \zeta^m = (s_{j_0})^{\ell_j-1}(\zeta^m).$$

Hence $s_j = (s_{j_0})^{\ell_j-1}$. \square

We also need the following simple lemma concerning a change of base.

LEMMA 8.5. *Let a_1, a_2 be generators of a free group F_2 . Let $E \subset F_2$ be the finite subset $E = \{a_1, a_2, a_2 a_1^{-1}, a_2 a_1^{-2}\}$. Put $E_- \stackrel{\text{def}}{=} \{a^{-1} : a \in E\}$. Suppose $\Psi : F_2 \rightarrow \mathcal{S}_3$ is a homomorphism of F_2 into the symmetric group \mathcal{S}_3 whose image is an Abelian subgroup of \mathcal{S}_3 which acts transitively on the set of three elements. Then there are generators $\mathbf{a}_1, \mathbf{a}_2 \in E \cup E_-$ such that $\Psi(\mathbf{a}_1)$ is a 3-cycle, $\Psi(\mathbf{a}_2) = \text{id}$ and the commutator $[\mathbf{a}_1, \mathbf{a}_2]$ is conjugate to $[a_1, a_2]$.*

Proof. By (the proof of) Lemma 8.4 the image of one of the original generators is a 3-cycle. If $\Psi(a_1)$ is a 3-cycle, then $\Psi(a_2 a_1^{-q})$ is the identity for q being either 0 or 1 or 2. (Recall that $\Psi(a_2)$ is a power of $\Psi(a_1)$ and $\Psi(a_1)^3 = \text{id}$.) Also,

$$[a_1, a_2 a_1^{-q}] = a_1 a_2 a_1^{-q} a_1^{-1} a_1^q a_2^{-1} = [a_1, a_2].$$

If $\Psi(a_1)$ is the identity then $\Psi(a_2)$ is a 3-cycle. For the pair $\mathbf{a}_1 = a_2, \mathbf{a}_2 = a_1^{-1}$ of generators the commutator $[\mathbf{a}_1, \mathbf{a}_2] = a_2 a_1^{-1} a_2^{-1} a_1$ is conjugate to the commutator $[a_1, a_2]$. \square

Let \mathcal{X} be a topological space and let a be an element of the fundamental group $\pi_1(\mathcal{X}, x_0)$ of \mathcal{X} with base point x_0 . Recall that a continuous map g from an annulus $A = \{z \in \mathbb{C} : r < |z| < R\}$ into \mathcal{X} is said to represent the conjugacy class \hat{a} of a if for some (and hence for any) $\rho \in (r, R)$ the map $g : \{|z| = \rho\} \rightarrow \mathcal{X}$ represents \hat{a} .

Let X be a torus with a disc removed. Its fundamental group $\pi_1(X, x_0)$ with base point x_0 is isomorphic to the free group F_2 with two generators.

LEMMA 8.6. *For each generator a of the fundamental group $\pi_1(X, x_0)$ of a torus with a disc removed there exists a conformal structure $w_a : X \rightarrow w_a(X) = Y_a$ of second kind with the following property. There exist two annuli A_a and A_∂ of conformal module strictly larger than $\eta = \frac{\pi}{2} \left(\log \frac{3+\sqrt{5}}{2} \right)^{-1}$, and two holomorphic mappings, $g_a : A_a \rightarrow X$ representing \hat{a} , and $g_\partial : A_\partial \rightarrow X$ representing the conjugacy class of the commutator of a standard pair of generators of $\pi_1(X, x_0)$.*

Without loss of generality we may assume that the mappings g_a and g_∂ are conformal mappings onto domains Ω_a and Ω_∂ , respectively, and that Ω_∂ has ∂X as one boundary component.

Proof. Let a' be another generator of $\pi_1(X, x_0)$ so that (a, a') forms a standard basis of $\pi_1(X, x_0)$. X is obtained from a closed torus X^c by removing a disc. The torus X^c can be written as \mathbb{C}/Λ for a lattice $\Lambda = \{k_1 \lambda_1 + k_2 \lambda_2 : k_1, k_2 \in \mathbb{Z}\}$. Here $\lambda_1, \lambda_2 \in \mathbb{C}$ are linearly independent over \mathbb{R} . The Riemann surface X may be covered by $\mathbb{C} \setminus (\Lambda + \rho \mathbb{D})$, where

$$\Lambda + \rho \mathbb{D} = \{z + \zeta : z \in \Lambda, |\zeta| < \rho\}$$

for a positive number ρ . We may assume that x_0 lifts to $\frac{1}{2}(\lambda_1 + \lambda_2)$ and that the conjugacy class of a is represented by the loop which lifts to the curve

$$t \rightarrow t\lambda_2 + \frac{1}{2}(\lambda_1 + \lambda_2), \quad t \in [0, 1],$$

and the conjugacy class of a' is represented by the loop that lifts to

$$t \rightarrow t\lambda_1 + \frac{1}{2}(\lambda_1 + \lambda_2), \quad t \in [0, 1].$$

Consider a self-homeomorphism \tilde{w}_a of \mathbb{C} such that

$$\tilde{w}_a(z + \lambda_1) = \tilde{w}_a(z) + \frac{\pi}{2} \frac{1}{\log \frac{3+\sqrt{5}}{2}} + 3\varepsilon$$

for a sufficiently small $\varepsilon > 0$ and

$$\tilde{w}_a(z + \lambda_2) = \tilde{w}_a(z) + i, \quad z \in \mathbb{C}.$$

Moreover, require that \tilde{w}_a maps 0 to 0 and maps $\rho\mathbb{D} = \{z \in \mathbb{C} : |z| < \rho\}$ into $\varepsilon\mathbb{D}$. The map \tilde{w}_a induces a map $\omega_a^\Lambda : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$,

$$\Lambda' = \left\{ k_1 \cdot \left(\frac{\pi}{2} \frac{1}{\log \frac{3+\sqrt{5}}{2}} + 3\varepsilon \right) + k_2 \cdot i : k_1, k_2 \in \mathbb{Z} \right\}.$$

If $\varepsilon > 0$ is small the restriction $\omega_a = \omega_a^\Lambda | X$ satisfies the requirements of the lemma. \square

THEOREM 8.1. *Suppose X is a torus with a disc removed. Let $a_1, a_2 \in \pi_1(X, x_0)$ be standard generators and let $E \subset \pi_1(X, x_0)$ be the set of Lemma 8.5 consisting of four elements. For each element $\mathbf{a} \in E$ we denote by $w_{\mathbf{a}}$ the conformal structure $w_{\mathbf{a}} : X \rightarrow Y_{\mathbf{a}} = w_{\mathbf{a}}(X)$ of second kind obtained in Lemma 8.6.*

Let f be a quasipolynomial of degree 3 on X which is isotopic to an algebroid function for each of the conformal structures $w_{\mathbf{a}}$, $\mathbf{a} \in E$. Then the isotopy class of f corresponds to the conjugacy class of a homomorphism

$$\Phi : \pi_1(X, x_0) \rightarrow \Gamma \subset \mathcal{B}_3$$

for a subgroup Γ of \mathcal{B}_3 . If f is irreducible, then Γ is generated by $\sigma_1 \sigma_2$ and the image $\Phi(\pi_1(X, x_0))$ contains elements other than powers of Δ_3^2 .

If f is reducible then Γ is generated by σ_1 and Δ_3^2 , or by $\sigma_1 \sigma_2 \sigma_1$.

In particular, in all cases the image $\Phi([a_1, a_2])$ of the commutator of the generators is the identity.

Proof. Let $\Phi : \pi_1(X, x_0) \rightarrow \mathcal{B}_3$ be a homomorphism whose conjugacy class corresponds to the isotopy class of f . Denote by $\Psi = \tau_3 \circ \Phi : \pi_1(X, x_0) \rightarrow \mathcal{S}_3$ the related homomorphism into the symmetric group. By the choice of the conformal structures and by Lemma 8.3 $\Psi([a_1, a_2]) = \text{id}$.

Let first f be irreducible. Then the subgroup $\Psi(\pi_1(X, x_0))$ of \mathcal{S}_3 acts transitively on the set of three points. Apply Lemma 8.5 to the free group $\pi_1(X, x_0)$ with generators a_1 and a_2 and to the homomorphism Ψ . We obtain new generators $\mathbf{a}_1, \mathbf{a}_2 \in E \cup E_-$ (with E and E_- being the sets of the lemma) such that $\Psi(\mathbf{a}_1)$ is a 3-cycle and $\Psi(\mathbf{a}_2) = \text{id}$. Put $s_1 = \Psi(\mathbf{a}_1)$, $s_2 = \Psi(\mathbf{a}_2)$, $b_1 = \Phi(\mathbf{a}_1)$, $b_2 = \Phi(\mathbf{a}_2)$. Then b_2 and $[b_1, b_2]$ are pure braids (the latter holds since $[\mathbf{a}_1, \mathbf{a}_2]$ is conjugate to $[a_1, a_2]$). Since f is isotopic to an algebroid function for $w_{\mathbf{a}_1}$ we obtain $\mathcal{M}(b_1) = \infty$

by Lemma 8.3. Since $\tau_3(b_1)$ is a 3-cycle, by the same lemma the braid b_1 must be conjugate to an integral power of $\sigma_1 \sigma_2$.

After conjugating Φ we may assume that $b_1 = (\sigma_1 \sigma_2)^{3\ell \pm 1} = (\sigma_1 \sigma_2)^{\pm 1} \cdot \Delta_3^{2\ell}$ for an integer ℓ . By Lemma 8.3 we also have $\mathcal{M}(b_2) = \infty$ for the pure braid b_2 . Hence, $b_2 = w^{-1} \sigma_1^{2k} \Delta_3^{2\ell'} w$ for integers k and ℓ' and a conjugating braid $w \in \mathcal{B}_3$. Also, $\mathcal{M}([b_1, b_2]) = \infty$ and $[b_1, b_2]$ is conjugate to $\sigma_1^{2k^*} \Delta_3^{2\ell^*}$ for integers k^* and ℓ^* .

LEMMA 8.7. *Let $b_1 = (\sigma_1 \sigma_2)^{\pm 1}$ and $b_2 = w^{-1} \sigma_1^{2k} w$ for an integer k and a braid $w \in \mathcal{B}_3$. If the commutator $[b_1, b_2]$ is conjugate to $\sigma_1^{2k^*} \Delta_3^{2\ell^*}$ for integers k^* and ℓ^* then b_2 is the identity and hence also the commutator is the identity.*

Proof of Lemma 8.7. Let $b \in \mathcal{B}_n$ be a pure braid (considered as an isotopy class of geometric braids with strands labeled). Define the linking number ℓ_{ij} of the i -th and j -th strand as follows. Discard all strands except the i -th and the j -th strand. We obtain a pure braid σ^{2m} . We call the integral number m the linking number between the two strands and denote it by ℓ_{ij} .

Note that the linking numbers \mathfrak{L}_{ij} of the braid $\sigma_1^{2k^*} \Delta_3^{2\ell^*}$ are equal to $\mathfrak{L}_{12} = k^* + \ell^*$, $\mathfrak{L}_{23} = \ell^*$, $\mathfrak{L}_{13} = \ell^*$. Since the braid is conjugate to a commutator its degree (the sum of exponents of generators in a representing word) must be zero. Hence, $2k^* + 6\ell^* = 0$. Hence, the (unordered) collection of linking numbers of $\sigma_1^{2k^*} \Delta_3^{2\ell^*}$ is $\{-2\ell^*, \ell^*, \ell^*\}$. Since conjugation only permutes linking numbers between pairs of strands of a pure braid, the unordered collection of linking numbers of pairs of strands of $[b_1, b_2]$ equals $\{-2\ell^*, \ell^*, \ell^*\}$ for the integer ℓ^* . For the pure braid σ_1^{2k} the linking number between the first and the second strand equals k , the linking numbers of the remaining pairs of strands equal zero.

For a braid $w \in \mathcal{B}_3$ let s_w be the permutation $s_w = \tau_3(w)$.

For a permutation $s \in \mathcal{S}_3$ acting on the set $\{1, 2, 3\}$ we denote as above by S the induced action on 3-tuples of numbers, $S(x_1, x_2, x_3) = (x_{s(1)}, x_{s(2)}, x_{s(3)})$. Order the linking numbers of pairs of strands of a braid $b \in \mathcal{B}_3$ as $(\ell_{23}, \ell_{13}, \ell_{12})$. Since

$$(s_w(1), s_w(2), s_w(3)) \xrightarrow{S_w^{-1}} (1, 2, 3) \xrightarrow{S_w} (s_w(1), s_w(2), s_w(3)),$$

the ordered tuple of linking numbers of pairs of strands of $w^{-1} b w$ equals

$$S_w((\ell_{23}, \ell_{13}, \ell_{12})) = (\ell_{s_w(2)s_w(3)}, \ell_{s_w(1)s_w(3)}, \ell_{s_w(1)s_w(2)}).$$

Hence, the ordered tuple of linking numbers between pairs of strands of $b_2 = w^{-1} \sigma_1^{2k} w$ equals $S_w(0, 0, k)$. Similarly, the ordered tuple of linking numbers between pairs of strands of b_2^{-1} is equal to $S_w(0, 0, -k)$.

Consider the commutator $b_1 b_2 b_1^{-1} b_2^{-1}$.

The ordered tuple of linking numbers of pairs of strands of the pure braid $b_1 b_2 b_1^{-1}$ equals $(S_1)^{-1} \circ S_w(0, 0, k)$ for the permutation $s_1 = \tau_3(b_1)$ (see figure 8.1).

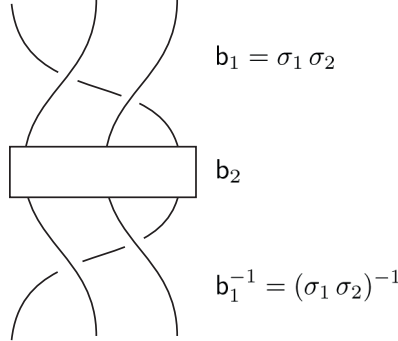


FIGURE 8.1

Hence the ordered tuple of linking numbers of pairs of strands of the commutator $b_1 b_2 b_1^{-1} \circ b_2^{-1}$ equals $S_w(0, 0, -k) + (S_1)^{-1} \circ S_w(0, 0, k)$. Since s_1 is a 3-cycle, the mapping $S_1(x_1, x_2, x_3) = (x_{s_1(1)}, x_{s_1(2)}, x_{s_1(3)})$ does not fix any pair of points among the x_1, x_2, x_3 (setwise). Hence the unordered 3-tuple of linking numbers of $b_1 b_2 b_1^{-1} b_2^{-1}$ is $\{k, -k, 0\}$. It can coincide with an unordered 3-tuple of the form $\{-2\ell^*, \ell^*, \ell^*\}$ only if $k = 0$. Hence b_2 is the identity and the commutator is the identity. \square

Since $\Delta_3^{2\ell}$ commutes with each 3-braid we have $[b_1, b_2] = [b_1, b_2]$ and hence $[b_1, b_2] = \text{id}$, $b_1 = (\sigma_1 \sigma_2)^{\pm 1} \cdot \Delta_3^{2\ell}$ and $b_2 = \Delta_3^{2\ell'}$.

This gives the statement of Theorem 8.1 for the case of **irreducible quasipolynomials**.

Let the quasipolynomial f be reducible on X . Let $\Phi : \pi_1(X, x_0) \rightarrow \mathcal{B}_3$ be a homomorphism whose conjugacy class corresponds to the isotopy class of f . For the homomorphism $\Psi = \tau_3 \circ \Phi : \pi_1(X, x_0) \rightarrow \mathcal{S}_3$ the image $\Psi(\pi_1(X, x_0))$ is generated either by a transposition (we may assume the transposition to be (12) by choosing the mapping Φ in its conjugacy class) or is equal to the identity in \mathcal{S}_3 . In the latter case f is solvable.

We may choose generators $a_1, a_2 \in E \cup E_-$ such that $[a_1, a_2]$ is conjugate to $[a_1, a_2]$ and $\Psi(a_2) = \text{id}$. Indeed, suppose $\Psi(\pi_1(X; x_0))$ is generated by the transposition (12) and neither $\Psi(a_1)$ nor $\Psi(a_2)$ is the identity. Then $\Psi(a_1) = \Psi(a_2) = (12)$, and $\Psi(a_2 a_1^{-1}) = \text{id}$. Hence as in the proof of Lemma 8.5 the generators a_1 and a_2 can be chosen as required.

Let $b_1 = \Phi(a_1)$, $b_2 = \Phi(a_2)$. By Lemma 8.3 we may assume, that b_1 is conjugate to either $\sigma_1^{k_1} \Delta_3^{2\ell_1}$ or to $(\sigma_1 \sigma_2 \sigma_1)^{k_1} = \Delta_3^{k_1}$ for integers k_1 and ℓ_1 , and b_2 is conjugate to either $\sigma_1^{2k_2} \Delta_3^{2\ell_2}$ for integers k_2 and ℓ_2 , or to an even power of Δ_3 . Conjugating Φ , we may assume that either $b_1 = B_1 \Delta_3^{2\ell_1}$ with $B_1 = \sigma_1^{k_1}$ or $B_1 = \sigma_2 \sigma_1^2 = \sigma_1^{-1}(\sigma_1 \sigma_2 \sigma_1) \sigma_1$, or $B_1 = \text{id}$. Respectively, b_2 is conjugate to either $\sigma_1^{2k_2} \Delta_3^{2\ell_1}$ or is equal to $\Delta_3^{2\ell_2}$. Since $[b_1, b_2]$ is a pure braid with infinite conformal module Lemma 8.3 implies that $[b_1, b_2]$ is conjugate to $\sigma_1^{2k^*} \Delta_3^{2\ell^*}$. Since the commutator has degree zero the equality $2k^* + 6\ell^* = 0$ holds and the unordered tuple of linking numbers equals $\{\ell^*, \ell^*, -2\ell^*\}$.

This implies that the commutator $[b_1, b_2]$ equals the identity. Indeed, if $b_2 = \Delta_3^{2\ell_2}$ this follows from the fact that Δ_3^2 is in the center of \mathcal{B}_3 . Suppose $b_2 = w^{-1} \sigma_1^{2k_2} w \Delta_3^{2\ell_1}$ for a braid $w \in \mathcal{B}_3$. The ordered tuple of linking numbers of pairs of strands of B_2 equals $S_w(k_2, 0, 0)$ and the respective ordered triple for B_2^{-1} is $S_w(-k_2, 0, 0)$. Here $s_w = \tau_3(w)$ and S_w is as before the generated action on 3-tuples of numbers. The ordered tuple of linking numbers of pairs of strands of $B_1 B_2 B_1^{-1}$ equals $(S_1)^{-1} \circ S_w(k_2, 0, 0)$, where $s_1 = \tau_3(B_1)$. For the unordered tuple of linking numbers of pairs of strands of $[b_1, b_2]$ we obtain either $\{k_2, -k_2, 0\}$ or $\{0, 0, 0\}$ (in dependence on s_w). Either of these unordered tuples can be equal to $\{\ell^*, \ell^*, -2\ell^*\}$ for some integer ℓ^* only if $\ell^* = 0$. We obtained that $[b_1, b_2] = \text{id}$ also in this case. The following lemma shows that either B_1 and B_2 are both powers of σ_1 , or they are both powers of $\sigma_2 \sigma_1^2$ and, hence, the theorem is proved. \square

LEMMA 8.8. *The centralizer of σ_1^k in B_3 , $k \neq 0$ an integral number, equals $\{\sigma_1^{k'} \Delta_3^{2\ell'} : k', \ell' \in \mathbb{Z}\}$.*

Proof. We use the standard homomorphism $\vartheta : \mathcal{B}_3 \rightarrow \text{SL}(2, \mathbb{Z})$ for which

$$\vartheta(\sigma_1) = \mathbf{A} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \vartheta(\sigma_2) = \mathbf{B} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Here $\text{SL}(2, \mathbb{Z})$ denotes the group of 2×2 matrices with determinant equal to 1. The kernel of ϑ is generated by $(\sigma_1 \sigma_2 \sigma_1)^4$. For $b \in \mathcal{B}_3$ we denote by $\langle b \rangle$ the subgroup of \mathcal{B}_3 generated by b . With this notation

$$\mathcal{B}_3 / \langle (\sigma_1 \sigma_2 \sigma_1)^4 \rangle$$

is isomorphic to $\text{SL}(2, \mathbb{Z})$. Notice that $\vartheta((\sigma_1 \sigma_2 \sigma_1)^2) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $b \in \mathcal{B}_3$ be in the centralizer of σ_1^k . Then $\mathbf{V} \stackrel{\text{def}}{=} \vartheta(b)$ is in the centralizer of \mathbf{A}^k , in other words,

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix},$$

i.e.

$$\begin{pmatrix} v_{11} + k v_{21} & v_{12} + k v_{22} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} v_{11} & k v_{11} + v_{12} \\ v_{21} & k v_{21} + v_{22} \end{pmatrix}.$$

Since $k \neq 0$ we have $v_{21} = 0$ and $v_{11} = v_{22}$. Since $\det \mathbf{V} = 1$ we obtain for an integer m

$$\mathbf{V} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \mathbf{V} = \begin{pmatrix} -1 & -m \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

Hence, either $\vartheta(b \sigma_1^{-m}) = \text{id}$ or $\vartheta(b \sigma_1^{-m} \Delta_3^2) = \text{id}$. Thus $b \sigma_1^{-m} = \Delta_3^{2\ell}$ for some integer ℓ . The lemma is proved. \square

Remark 8.1. D.Calegari and A.Walker gave the following example of two elements of the braid group \mathcal{B}_3 with non-trivial commutator of zero entropy. Put $b_1 = (\sigma_2)^{-1} \sigma_1$, $b_2 = \sigma_2 (\sigma_1)^{-1}$. The commutator equals $[b_1, b_2] = (\sigma_2)^{-6} \Delta_3^2$. Indeed,

$$\begin{aligned} b_1^{-1} b_2^{-1} &= (\sigma_1)^{-1} \sigma_2 \sigma_1 (\sigma_2)^{-1} = (\sigma_1)^{-1} \sigma_2 \sigma_1 \sigma_2 (\sigma_2)^{-2} = (\sigma_1)^{-1} \sigma_1 \sigma_2 \sigma_1 (\sigma_2)^{-2} \\ &= \sigma_2 \sigma_1 (\sigma_2)^{-2}, \\ b_1 b_2 &= (\sigma_2)^{-1} \sigma_1 \sigma_2 (\sigma_1)^{-1} = (\sigma_2)^{-2} \sigma_2 \sigma_1 \sigma_2 (\sigma_1)^{-1} = (\sigma_2)^{-2} \sigma_1 \sigma_2 \sigma_1 (\sigma_1)^{-1} \\ &= (\sigma_2)^{-2} \sigma_1 \sigma_2. \end{aligned}$$

Hence,

$$\begin{aligned} [b_1, b_2] &= b_1 b_2 b_1^{-1} b_2^{-1} = (\sigma_2)^{-3} \sigma_2 \sigma_1 \sigma_2 \cdot \sigma_2 \sigma_1 \sigma_2 (\sigma_2)^{-3} = (\sigma_2)^{-3} \Delta_3^2 (\sigma_2)^{-3} \\ &= (\sigma_2)^{-6} \Delta_3^2. \end{aligned}$$

It is clear that the commutator $[b_1, b_2]$ has entropy zero.

PROBLEM 8.1. *Can two braids in \mathcal{B}_3 of zero entropy have a non-trivial commutator of zero entropy?*

(If not, Theorem 8.1 would hold with two instead of four conformal structures.) The proof of Theorem 8.1 gives the following corollary which is weaker than a negative answer to Problem 8.1.

COROLLARY 8.1. *Let b_1 and b_2 be braids in \mathcal{B}_3 . If one of the braids is pure or*

$$h(b_1) = h(b_2) = h([b_1, b_2]) = h(b_2 b_1^{-1}) = h(b_2 b_1^{-2}) = 0$$

then $[b_1, b_2] = \text{id}$.

THEOREM 8.2. *Let $n \geq 2$ be a natural number and let $\Gamma \subset \mathcal{B}_n$ be a subgroup generated by a periodic braid $b \in \mathcal{B}_n$.*

Consider a separable quasipolynomial of degree n on a punctured torus X whose isotopy class corresponds to the equivalence class of a homomorphism $\Phi : \pi_1(X, z_0) \rightarrow \Gamma$.

Then the following holds.

- (1) *The quasipolynomial is isotopic to an algebroid function f on X .*
- (2) *The zero set*

$$\mathfrak{S}_f = \{(z, \zeta) \in X \times \mathbb{C} : f(z, \zeta) = 0\}$$

extends to a leaf (or union of leaves) of a (non-singular) holomorphic foliation of the total space V of a holomorphic line bundle $V \rightarrow X^{\text{Cl}}$ over the closed torus X^{Cl} . The line bundle is smoothly trivial.

Proof. The group Γ is Abelian. The fundamental group $\pi_1(X^{\text{Cl}}, z_0)$ of the closed torus X^{Cl} is the abelianization of $\pi_1(X, z_0)$. Hence Φ defines a homomorphism from $\pi_1(X^{\text{Cl}}, z_0)$ to Γ which we also denote by Φ .

Let (a_1, a_2) be a basis of $\pi_1(X^{\text{Cl}}, z_0)$. Then $\Phi(a_j) = b^{k_j}$, $j = 1, 2$, for the generator b of Γ and integral numbers k_j . We may assume that the k_j are relatively prime by taking otherwise a subgroup of Γ . We may choose another basis (a_1, a_2) of $\pi_1(X^{\text{Cl}}, z_0)$ so that $\Phi(a_1) = b$ and $\Phi(a_2) = \text{id}$. Indeed, there are integers ℓ_1, ℓ_2 such that $\ell_1 k_1 + \ell_2 k_2 = 1$. Put $a_1 = a_1^{\ell_1} \cdot a_2^{\ell_2}$ and $a_2 = a_1^{-k_2} \cdot a_2^{k_1}$. Then $\Phi(a_1) = b^{\ell_1 k_1 + \ell_2 k_2} = b$ and $\Phi(a_2) = \text{id}$. Also, $a_1 = a_1^{k_1} a_2^{-\ell_2}$ and $a_2 = a_1^{k_2} a_2^{\ell_1}$, so that (a_1, a_2) is a basis.

Consider the mapping class \mathbf{m}_b in the mapping class group $\mathfrak{M}(\overline{\mathbb{D}}; \partial \mathbb{D}, E_n^0)$ associated to b . Let $\mathcal{H}_\infty(\mathbf{m}_b)$ be its image in $\mathfrak{M}(\mathbb{P}^1; \infty, E_n^0)$. Recall that the kernel of the mapping $\mathbf{m}_b \rightarrow \mathcal{H}_\infty(\mathbf{m}_b)$ is equal to the center of the braid group \mathcal{B}_n which is generated by Δ_n^2 . Conjugating Γ we may assume that the mapping class $\mathcal{H}_\infty(\mathbf{m}_b)$ is represented by multiplication by a primitive root of unity considered as homeomorphism of \mathbb{C} with set of distinguished points E_n . By changing the generator of Γ if necessary, we may assume that

- (1) either $\omega = e^{\frac{2\pi i}{n}}$ and E_n consists of n equidistributed points on a circle with center zero, or

- (2) $\omega = \frac{2\pi i}{n-1}$ and E_n consists of the origin and $n-1$ equidistributed points on a circle with center zero.

Consider the homomorphism from Γ to the group of complex linear transformations of \mathbb{C} , which assigns to the generator $b \in \Gamma$ the linear transformation $\zeta \rightarrow \omega \cdot \zeta$, $\zeta \in \mathbb{C}$. The kernel of this homomorphism is the center of the braid group \mathcal{B}_n . The center of \mathcal{B}_n is a subgroup of Γ of finite index (either n in case (1) or $n-1$ in case (2)). Composing Φ with this homomorphism we obtain a homomorphism Ξ from $\pi_1(X^{\text{Cl}}, x_0)$ to the group of linear transformations of \mathbb{C} .

Represent the closed torus X^{Cl} as quotient $X^{\text{Cl}} = \mathbb{C}/\Lambda$ for a lattice $\Lambda = \{\lambda_1 k_1 + \lambda_2 k_2 : k_1, k_2 \in \mathbb{Z}\}$ with real linearly independent numbers $\lambda_1, \lambda_2 \in \mathbb{C}$. Denote by p_1 the covering map $p_1 : \mathbb{C} \rightarrow \mathbb{C}/\Lambda$. Identify the fundamental group $\pi_1(\mathbb{C}/\Lambda, z_0) = \pi_1(X^{\text{Cl}}, z_0)$ with the group of covering translations which we also denote by Λ .

The fundamental group Λ of $X^{\text{Cl}} \cong \mathbb{C}/\Lambda$ acts on $\mathbb{C} \times \mathbb{P}^1$ as follows. Associate to $\lambda \in \Lambda$ the mapping

$$(8.4) \quad (z, \zeta) \xrightarrow{\lambda} (\lambda(z), \Xi(\lambda)(\zeta)), \quad (z, \zeta) \in \mathbb{C} \times \mathbb{P}^1.$$

The action is free and properly discontinuous. Hence the quotient $(\mathbb{C} \times \mathbb{P}^1)/\Lambda$ is a complex manifold. For each element of the quotient its projection to \mathbb{C}/Λ is well-defined:

$$\begin{array}{ccc} (z, \zeta) & \sim & (\lambda(z), \Xi(\lambda)(\zeta)) \\ \downarrow & & \downarrow \\ z & \sim & \lambda(z) \end{array}$$

The equivalence relation in the upper line gives the quotient $(\mathbb{C} \times \mathbb{P}^1)/\Lambda$, the relation in the lower line gives \mathbb{C}/Λ . The fiber of the projection $p : (\mathbb{C} \times \mathbb{P}^1)/\Lambda \rightarrow \mathbb{C}/\Lambda$ equals \mathbb{P}^1 and the transition functions are complex linear in each fiber. Hence

$$(8.5) \quad (\mathbb{C} \times \mathbb{P}^1)/\Lambda \xrightarrow{p} \mathbb{C}/\Lambda$$

is a holomorphic line bundle over \mathbb{C}/Λ . The trivial holomorphic foliation on $\mathbb{C} \times \mathbb{P}^1$ with leaves $\mathbb{C} \times \{\zeta\}$, $\zeta \in \mathbb{P}^1$, induces a holomorphic foliation on $(\mathbb{C} \times \mathbb{P}^1)/\Lambda$ (since $\lambda \in \Lambda$ maps leaves of the trivial foliation on $\mathbb{C} \times \mathbb{P}^1$ to leaves). The foliation is transversal to the fibers.

Restrict the line bundle to the punctured torus $X \cong \mathbb{C} \setminus \Lambda / \Lambda$. The restriction is holomorphically trivial (see, e.g. [13], Theorem 30.3). In other words, there is a biholomorphic mapping

$$(8.6) \quad \mathcal{I} : ((\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1)/\Lambda \rightarrow ((\mathbb{C} \setminus \Lambda)/\Lambda) \times \mathbb{P}^1$$

which preserves fibers and is complex linear on each fiber.

Consider the set E_n which we associated to the braid b (which depends on whether case (1) or (2) holds for b). The set $(\mathbb{C} \setminus \Lambda) \times E_n$ is saturated (i.e. it equals the union of leaves) for the trivial foliation of $(\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1$. The covering map

$$(8.7) \quad (\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1 \rightarrow ((\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1)/\Lambda$$

maps this set to a relatively closed saturated set \mathfrak{S}_Λ in the induced foliation on the quotient. The set \mathfrak{S}_Λ intersects each fiber along n pairwise distinct points. Notice that in case (1) the set \mathfrak{S}_Λ in the quotient consists of a single leaf, while in case

(2) it is the union of two leaves. Denote the image $\mathcal{I}(\mathfrak{S}_\Lambda)$ of the relatively closed set \mathfrak{S}_Λ under \mathcal{I} by $\tilde{\mathfrak{S}}$. The set $\tilde{\mathfrak{S}}$ is a complex curve in

$$((\mathbb{C} \setminus \Lambda) / \Lambda) \times \mathbb{P}^1 \cong X \times \mathbb{P}^1$$

which intersects each fiber along n pairwise distinct points. Hence $\tilde{\mathfrak{S}}$ equals the zero set $\mathfrak{S}_{\tilde{f}}$ of an algebroid function \tilde{f} of degree n on X .

We will show now that the free isotopy class of \tilde{f} corresponds to the conjugacy class of a homomorphism $\tilde{\Phi}$, such that

$$(8.8) \quad \tilde{\Phi}(\mathbf{a}_1) = \Delta_3^{2\ell_1} b, \quad \tilde{\Phi}(\mathbf{a}_2) = \Delta_3^{2\ell_2}$$

for integers ℓ_1 and ℓ_2 . Consider a simple closed loop $\gamma : [0, 1] \rightarrow X = \mathbb{C} \setminus \Lambda / \Lambda$ with base point $\gamma(0) = \gamma(1) = z_0$. We will choose γ so that it represents an element $a_\gamma \in \pi_1(\mathbb{C} \setminus \Gamma)$ which equals either the element \mathbf{a}_1 or \mathbf{a}_2 of $\pi_1(X, x_0)$. Restrict the bundle $((\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1) / \Lambda$ to the loop $\gamma([0, 1])$ and denote by the same letter \mathcal{I} the restricted bundle isomorphism. Consider the inverse mapping

$$(8.9) \quad \mathcal{I}^{-1} : \gamma([0, 1]) \times \mathbb{P}^1 \rightarrow p^{-1}(\gamma([0, 1])) \subset ((\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1) / \Lambda.$$

The mapping \mathcal{I}^{-1} maps each point (z, ζ) in $(\mathbb{C} \setminus \Lambda / \Lambda) \times \mathbb{P}^1$ to a point in $(\mathbb{C} \setminus \Lambda \times \mathbb{P}^1) / \Lambda$. Choose local coordinates on the total space $(\mathbb{C} \setminus \Lambda / \Lambda) \times \mathbb{P}^1$ of the trivial bundle $(\mathbb{C} \setminus \Lambda / \Lambda) \times \mathbb{P}^1 \rightarrow \mathbb{C} \setminus \Lambda / \Lambda$ by choosing a local lift $(\tilde{z}, \zeta) \in \mathbb{C} \setminus \Lambda \times \mathbb{P}^1$ of (z, ζ) . Consider the lift of $\mathcal{I}^{-1}(z, \zeta)$ with first coordinate (\tilde{z}, ζ) . It defines local coordinates on the total space $(\mathbb{C} \setminus \Lambda \times \mathbb{P}^1) / \Lambda$ of the bundle (8.7). Since \mathcal{I} is a bundle isomorphism the lift of $\mathcal{I}^{-1}(z, \zeta)$ has the form $(\tilde{z}, \psi_{\tilde{z}}^{-1}(\zeta))$, where $\psi_{\tilde{z}}$ is a complex linear homeomorphism of \mathbb{P}^1 onto itself. The following equation holds for $\lambda \in \Lambda$

$$(8.10) \quad \psi_{\lambda(\tilde{z})}(\zeta) = \psi_{\tilde{z}}(\Xi(\lambda)^{-1}(\zeta)), \quad \zeta \in \mathbb{P}^1.$$

Indeed, $\mathcal{I}^{-1}(z, \zeta)$ is represented by $(\tilde{z}, \psi_{\tilde{z}}^{-1}(\zeta))$ and also by $(\lambda(\tilde{z}), \Xi(\lambda)(\psi_{\tilde{z}}^{-1}(\zeta)))$. Choosing lifts with first coordinate equal to $\lambda(\tilde{z})$ we see that $\mathcal{I}^{-1}(z, \zeta)$ is also represented by $(\lambda(\tilde{z}), \psi_{\lambda(\tilde{z})}^{-1}(\zeta))$. Hence,

$$(8.11) \quad (\lambda(\tilde{z}), \Xi(\lambda)(\psi_{\tilde{z}}^{-1}(\zeta))) = (\lambda(\tilde{z}), \psi_{\lambda(\tilde{z})}^{-1}(\zeta)).$$

(8.10) follows. Denote by $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{C} \setminus \Lambda$ the lift of γ to the covering $\mathbb{C} \setminus \Lambda$ with initial point \tilde{z}_0 , $p_1(\tilde{z}_0) = z_0$.

Consider the mappings $\psi_{\tilde{\gamma}(t)}$ for $t \in [0, 1]$. Thus, the equivalence class of $(\tilde{\gamma}(t), \xi)$, $\xi \in \mathbb{P}^1$, is mapped by \mathcal{I} to the point

$$(8.12) \quad (\gamma(t), \psi_{\tilde{\gamma}(t)}(\xi)) \in ((\mathbb{C} \setminus \Lambda) / \Lambda) \times \mathbb{P}^1.$$

We may assume that $\psi_{\tilde{\gamma}(1)} = \text{id}$ after composing \mathcal{I} with a self-map of $((\mathbb{C} \setminus \Lambda) / \Lambda) \times \mathbb{P}^1$ depending only on the \mathbb{P}^1 -coordinate of the points. Consider the zero set $\mathfrak{S}_{\tilde{f}}$ of the quasipolynomial \tilde{f} . Its part over the loop $\gamma([0, 1])$ is equal to the image

$$(8.13) \quad \mathcal{I} \left(\left(\bigcup_{t \in [0, 1]} \{\tilde{\gamma}(t)\} \times E_n \right) / \Lambda \right) = \bigcup_{t \in [0, 1]} (\{\gamma(t)\} \times \psi_{\tilde{\gamma}(t)}(E_n)).$$

Notice that

$$(8.14) \quad (\{\tilde{\gamma}(0)\} \times E_n) / \Lambda = (\{\tilde{\gamma}(1)\} \times \Xi(a_\gamma)(E_n)) / \Lambda = (\{\tilde{\gamma}(1)\} \times E_n) / \Lambda$$

since E_n is invariant under multiplication by the number ω associated to the braid b . Hence, $\psi_{\tilde{\gamma}(0)}(E_n) = \psi_{\tilde{\gamma}(1)}(E_n)$ by (8.10), and the mapping

$$(8.15) \quad [0, 1] \ni t \rightarrow \psi_{\tilde{\gamma}(t)}(E_n) \in C_n(\mathbb{C})/\mathcal{S}_n$$

defines a geometric braid with base point E_n . Equality (8.13) relates this braid to the zero set of the quasipolynomial \tilde{f} . It shows that the geometric braid defines the monodromy of the quasipolynomial \tilde{f} along γ . By (8.10) we have for the element $a_\gamma \in \Lambda$ represented by γ

$$(8.16) \quad \psi_{\tilde{\gamma}(1)}(\zeta) = \psi_{\tilde{\gamma}(0)}(\Xi(a_{\tilde{\gamma}(0)})^{-1}(\zeta)), \quad \zeta \in \mathbb{P}^1.$$

Since $\psi_{\tilde{\gamma}(1)} = \text{id}$, we obtain

$$(8.17) \quad \psi_{\tilde{\gamma}(0)} = \Xi(a_\gamma).$$

Consider an isotopy of the family $\psi_{\tilde{\gamma}(t)}$, $t \in [0, 1]$, to a continuous family $\psi_{\tilde{\gamma}(t)}^{\text{rel}}$, $t \in [0, 1]$, of self-homeomorphisms of \mathbb{P}^1 which fix the complement of a large disc $R_1 \mathbb{D}$ so that $\psi_{\tilde{\gamma}(t)}^{\text{rel}} = \psi_{\tilde{\gamma}(t)}$ on a disc $R \mathbb{D}$, $R < R_1$ containing $\bigcup_{t \in [0, 1]} \psi_{\tilde{\gamma}(t)}(E_n)$. We

may assume that $\psi_{\tilde{\gamma}(1)}^{\text{rel}} = \text{id}$. The family $\psi_{\tilde{\gamma}(t)}^{\text{rel}}$, $t \in [0, 1]$, is a parametrizing isotopy for the geometric braid (8.15) (in $[0, 1] \times R_1 \mathbb{D}$).

The homeomorphisms $\psi_{\tilde{\gamma}(0)}^{\text{rel}}$ and $\psi_{\tilde{\gamma}(0)}$ represent the same mapping class in $\mathfrak{M}(\mathbb{P}^1; \infty, E_n)$. This mapping class is also represented by $\Xi(a_\gamma) \in \mathcal{H}_\infty(\mathfrak{m}_b)$. Hence, in case γ represents \mathbf{a}_1 the isotopy class of the geometric braid $\psi_{\tilde{\gamma}(t)}(E_n)$ equals

$$(8.18) \quad \tilde{\Phi}(\mathbf{a}_1) = b \cdot \Delta_3^{2\ell_1} = \Phi(\mathbf{a}_1) \cdot \Delta_3^{2\ell_1} \text{ for an integer } \ell_1.$$

In case γ represents \mathbf{a}_2 , the isotopy class of $\psi_{\tilde{\gamma}(t)}(E_n)$ equals

$$(8.19) \quad \tilde{\Phi}(a_\lambda) = \Delta_3^{2\ell_2} = \Phi(a_2) \cdot \Delta_3^{2\ell_2}$$

for an integer ℓ_2 .

To obtain a quasipolynomial f whose free isotopy class corresponds to the conjugacy class of the homomorphism Φ rather than to $\tilde{\Phi}$ we change the trivialization of the line bundle

$$((\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1)/\Lambda \rightarrow \mathbb{C} \setminus \Lambda/\Lambda.$$

For this we consider the map

$$(8.20) \quad (\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1 \ni (x, \zeta) \rightarrow (x, e^{2\pi i F(x)} \zeta) \in (\mathbb{C} \setminus \Lambda) \times \mathbb{P}^1,$$

where F is a holomorphic function on $\mathbb{C} \setminus \Lambda$ such that

$$(8.21) \quad F(x + \lambda_1) = \ell_1 \quad \text{and} \quad F(x + \lambda_2) = \ell_2.$$

Here $\Lambda = \{\lambda_1 n_1 + \lambda_2 n_2, n_1, n_2 \in \mathbb{Z}\}$, and λ_1 corresponds to $a_1 \in \pi_1(\mathbb{C}/\Lambda, x_0)$, λ_2 corresponds to $a_2 \in \pi_1(\mathbb{C}/\Lambda, x_0)$. The map (8.20) descends to an isomorphism \mathcal{I}_1 of the line bundle $((\mathbb{C} \setminus \Lambda)/\Lambda) \times \mathbb{P}^1$ to itself. Consider the composition $\mathcal{I} \circ \mathcal{I}_1$ of \mathcal{I} with this isomorphism. This gives the required trivialization.

A holomorphic function F on $\mathbb{C} \setminus \Lambda$ satisfying (8.21) exists for all $\ell_1, \ell_2 \in \mathbb{Z}$.

An elementary way to see this for $\ell_2 = 0$ and any $\ell_1 \in \mathbb{Z}$ is the following. Let $\chi(t_1 \lambda_1 + t_2 \lambda_2)$, $(t_1, t_2) \in \mathbb{R}^2$, be a smooth function depending only on t_1 which satisfies the equation $\chi((t_1 + 1) \lambda_1) = \lambda_1 + \chi(t_1 \lambda_1)$, $t_1 \in \mathbb{R}$, and is constant in a neighbourhood of $t_1 = 0$. The 1-form $\bar{\partial} \chi$ descends to a smooth 1-form ω on the torus \mathbb{C}/Λ . Let g be a solution of the equation $\bar{\partial} g = \omega$ on the punctured torus $\mathbb{C} \setminus \Lambda/\Lambda$. Denote by the same letter g the lift of g to $\mathbb{C} \setminus \Lambda$. Put $F = \chi - g$.

Part (1) of the theorem is proved.

The set $\mathfrak{S}_\Lambda = (\mathcal{I} \circ \mathcal{I}_1)^{-1}(\mathfrak{S}_f)$ extends to the saturated set $\mathfrak{S} = (\mathbb{C} \times E_n)/\Lambda$ in the holomorphic foliation of $(\mathbb{C} \times \mathbb{P}^1)/\Lambda$. This proves the first statement of (2).

The bundle $(\mathbb{C} \times \mathbb{P}^1)/\Lambda$ is trivial as a smooth bundle. Indeed, $\Phi(\mathbf{a}_2) = \text{id}$, and the triviality of the bundle as a smooth bundle reduces to the triviality of a smooth line bundle over a circle. The theorem is proved. \square

Proof of Theorem 2. Choose the four conformal structures of Theorem 8.1. If the separable quasipolynomial f of degree 3 is **irreducible** and isotopic to an algebroid function for each of the four conformal structures then Theorem 8.2 applies. We obtain (1) and (3). Since the holomorphic line bundle is smoothly trivial and the monodromy of the quasipolynomial around the puncture is the identity, the quasipolynomial smoothly extends to the closed torus. This proves (2) for the irreducible case.

If a separable quasipolynomial f of degree 3 is **reducible** and isotopic to an algebroid function for each of the four conformal structures, then by Theorem 8.1 its isotopy class corresponds to the conjugacy class of a homomorphism of $\pi_1(X, x_0)$ onto the subgroup Γ of \mathcal{B}_3 generated by σ_1 and Δ_3^2 , or onto the subgroup Γ generated by $\sigma_1 \sigma_2 \sigma_1$. The braid $\sigma_1 \sigma_2 \sigma_1$ is periodic. Hence, in the second case Theorem 8.1 applies and Theorem 2 follows in the same way as in the case of irreducible quasipolynomials.

Consider the first case. We may assume that for generators \mathbf{a}_1 and \mathbf{a}_2 of $\pi_1(X, x_0)$ we have $\Phi(\mathbf{a}_1) = \sigma_1^k \Delta_3^{2\ell}$ and $\Phi(\mathbf{a}_2) = \sigma_1^{2k'} \cdot \Delta_3^{2\ell'}$.

Consider the homomorphism $\Phi_1 : \pi_1(X, x_0) \rightarrow \mathcal{B}_2$ into the braid group on two strands for which $\Phi_1(\mathbf{a}_1) = \sigma^k$ and $\Phi_1(\mathbf{a}_2) = \sigma^{2k'}$. Theorem 8.2 applies to this situation. For each conformal structure on X , including conformal structures of first kind, we obtain an algebroid function f_1 whose free isotopy class corresponds to the conjugacy class of Φ_1 .

Take any conformal structure $w : X \rightarrow w(X)$ of second kind on X . The Riemann surface $Y = w(X)$ may be considered as relatively compact subset of a Riemann surface $w_1(X)$ for a conformal structure w_1 of first kind on X . Denote by $Y^c = (w_1(X))^{\text{cl}}$ the closed Riemann surface containing Y . Take the algebroid function $f_1 : w_1(X) \rightarrow \mathfrak{P}_2 \cong C_2(\mathbb{C})/\mathcal{S}_2$ constructed for $w_1(X)$ and restrict it to Y . The image $f_1(Y)$ of the relatively compact subset Y of $w_1(X)$ is contained in a bounded subset $C_2(R\mathbb{D})/\mathcal{S}_2$ of the symmetrized configuration space $C_2(\mathbb{C})/\mathcal{S}_2$. (R is a large positive number.) By an abuse of notation we write

$$f_1(y, \zeta) = \zeta^2 + a_1(y)\zeta + a_0(y), \quad \zeta \in \mathbb{C}, \quad y \in Y = w(X),$$

where $\zeta \rightarrow f_1(y, \zeta)$ is the monic polynomial corresponding to $f_1(y) \in \mathfrak{P}_2 \cong C_2(\mathbb{C})/\mathcal{S}_2$.

Let $R' > R$ be a positive number. Put

$$\tilde{f}(y, \zeta) = f_1(y, \zeta) \cdot (R' - \zeta), \quad \zeta \in \mathbb{C}, \quad y \in Y.$$

\tilde{f} is a separable algebroid function on Y whose free isotopy class corresponds to the conjugacy class of the homomorphism $\tilde{\Phi} : \pi_1(X, x_0) \rightarrow \mathcal{B}_3$, for which $\tilde{\Phi}(\mathbf{a}_1) = \sigma_1^k$, $\tilde{\Phi}(\mathbf{a}_2) = \sigma_1^{2k'}$.

By changing the trivialisation of the line bundle over Y we obtain an algebroid function f on Y with $f \circ w^{-1}$ isotopic to the original quasipolynomial. We obtained

(1) for the reducible case. Part (2) is obtained in the same way as in the irreducible case.

It remains to show part (4). Suppose we have a separable algebroid function on a punctured torus X whose free isotopy class corresponds to the conjugacy class of the homomorphism $\Phi : \pi_1(X, x_0) \rightarrow \mathcal{B}_3$ with $\Phi(a_1) = \sigma_1^k \Delta_3^{2\ell}$, $\Phi(a_2) = \sigma_1^{2k'} \Delta_3^{2\ell'}$ for integers k, ℓ, k' and ℓ' and generators a_1, a_2 of $\pi_1(X, x_0)$. Let X^{Cl} be the one-point compactification of X . Consider the unramified double cover $\widetilde{X^{\text{Cl}}}$ of X^{Cl} which has generators \tilde{a}_1, \tilde{a}_2 of the fundamental group $\pi_1(\widetilde{X^{\text{Cl}}}, \tilde{x}_0)$ corresponding to a_1^2 and a_2 . X is doubly covered by a twice punctured torus \tilde{X} . Lift the quasipolynomial f to a quasipolynomial \tilde{f} on \tilde{X} . The isotopy class of \tilde{f} corresponds to the conjugacy class of a homomorphism $\tilde{\Phi} : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \mathcal{B}_3$ with $\tilde{\Phi}(\tilde{a}_1) = \sigma_1^{2k} \cdot \Delta_3^{4\ell}$ and $\tilde{\Phi}(\tilde{a}_2) = \sigma_1^{2k'} \cdot \Delta_3^{2\ell'}$. Moreover, the monodromy of \tilde{f} around each of the two punctures is the identity since each corresponds to the commutator of some powers of σ_1 . Thus, all monodromies of \tilde{f} on \tilde{X} correspond to pure braids. We obtained that \tilde{f} lifts to a holomorphic mapping from \tilde{X} to $C_3(\mathbb{C})$. Hence, we obtain three holomorphic functions $z_1(\tilde{x})$, $z_2(\tilde{x})$, $z_3(\tilde{x})$, $\tilde{x} \in \tilde{X}$. Consider for each \tilde{x} the complex affine self-map $\mathbf{a}(\tilde{x})$ of \mathbb{C} ,

$$\mathbf{a}(\tilde{x})(\zeta) = \frac{\zeta - z_1(\tilde{x})}{z_2(\tilde{x}) - z_1(\tilde{x})}.$$

The family $\mathbf{a}(\tilde{x})$ depends holomorphically on \tilde{x} and for each \tilde{x} we have $\mathbf{a}(\tilde{x})(z_1(\tilde{x})) = 0$, $\mathbf{a}(\tilde{x})(z_2(\tilde{x})) = 1$. We obtain a holomorphic function $Z_3(\tilde{x}) = \mathbf{a}(\tilde{x})(z_3(\tilde{x}))$, $\tilde{x} \in \tilde{X}$, with values in $\mathbb{C} \setminus \{0, 1\}$. The index of Z_3 and of $Z_3 - 1$ around each of the punctures of \tilde{X} is zero.

Hence the restriction of Z_3 to a punctured disc around each of the two punctures of \tilde{X} lifts to a holomorphic map of the punctured disc to the upper half-plane \mathbb{C}_+ , the universal covering of $\mathbb{C} \setminus \{0, 1\}$. Thus, the lift extends to a holomorphic function on a disc around each puncture. Hence Z_3 extends to a holomorphic function of the closed torus $\widetilde{X^{\text{Cl}}}$ with values in $\mathbb{C} \setminus \{0, 1\}$. Lift the extended function to a holomorphic function from \mathbb{C} , the universal covering of $\widetilde{X^{\text{Cl}}}$, to \mathbb{C}_+ , the universal covering of $\mathbb{C} \setminus \{0, 1\}$. The lift must be constant, hence Z_3 is constant. This means that $k = k' = 0$ and the algebroid function is isotrivial. Theorem 2 is proved. \square

Remark 8.2. Statement (2) of Theorem 2 is not true in general for braid groups on more than three strands. Moreover, in general, the commutator of two braids of zero entropy may be non-trivial and of zero entropy, even if one of the braids is pure. Indeed, let X be a torus with a disc removed. Consider, for example, a separable quasipolynomial of degree four on X whose isotopy class corresponds to the conjugacy class of the homomorphism $\Phi : \pi_1(X, x_0) \rightarrow \mathcal{B}_4$ with $\Phi(a_1) = b_1 = \sigma_1^2$, $\Phi(a_2) = b_2 = (\sigma_2)^{-1} (\sigma_3)^{-1} (\sigma_1)^{-1} (\sigma_2)^{-1}$ (see figure 8.2a) for generators a_1, a_2 of $\pi_1(X, x_0)$.

A similar argument as in the proof of Theorem 2 (in the case of a reducible quasipolynomial of degree 3) shows that the quasipolynomial is isotopic to an algebroid function for each conformal structure of second kind on X . But the commutator $[b_1, b_2] = \sigma_1^2 \cdot \sigma_3^{-2} \neq \text{id}$ (see figure 8.2b).

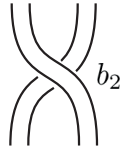


Figure 8.2 a

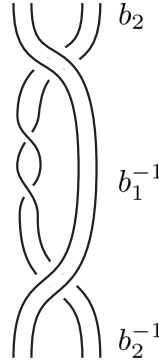


Figure 8.2 b

Hence, the quasipolynomial does not extend to a smooth quasipolynomial on the closed torus. The following problem arises.

Problem 8.2. *What is the analogue of Corollary 8.1 for braid groups on more than three strands?*

Remark 8.3. For the proof of Theorem 2 the Theorem 1 is not needed. However, Theorem 1 together with known results on entropy give the explicit information on the conformal structures which may serve in Theorem 8.1. This allows to give explicit information about suitable choices of the four conformal structures used in Theorem 2.

We will now discuss results of Gorin, Lin and Zjuzin. Gorin and Lin initiated a series of papers related to reducibility and solvability of quasipolynomials on complex manifolds or on more general topological spaces. A particular question was the comparison of conditions for solvability (resp. reducibility) of all quasipolynomials in a class, respectively of all algebroid functions in this class. In one of the papers they considered separable monic polynomials of degree n with coefficients in commutative Banach algebras and asked whether solvability of all such polynomials implies solvability of all separable quasipolynomials of degree n on the space of maximal ideals of the algebra. They gave a negative answer by finding conjugacy classes of braids with finite conformal module.

We will discuss here the result of [15]. For simplicity we will restrict ourselves to the case when the degree of the quasipolynomial is a prime number. (The general case has been considered by Lin and later by Zjuzin. An alternative proof of the general case can be given as well.) In [15] Gorin and Lin proved that for each prime number n there exists a number r_n such that any separable algebroid function of degree n on an annulus A of conformal module strictly larger than r_n is reducible provided the index of its discriminant $z \rightarrow D_n(f_z)$, $z \in A$, is divisible by n . Recall that the value at a point $z \in A$ of a separable algebroid function of degree n on A is a polynomial $f_z \in \mathfrak{P}_n$. $D_n(f_z)$ denotes its discriminant. Recall that D_n is a function on the space of all polynomials $\overline{\mathfrak{P}}_n$ of degree n which vanishes exactly on the set of polynomials with multiple zeros. Explicitly, for a polynomial $p \in \overline{\mathfrak{P}}_n$,

$\mathbf{p}(\zeta) = \prod_{j=1}^n (\zeta - \zeta_j)$, its discriminant equals

$$D_n(\mathbf{p}) = \prod_{i < j} (\zeta_i - \zeta_j)^2.$$

The index of the mapping

$$A \ni z \rightarrow D_n(f_z) \in \mathbb{C} \setminus \{0\}, \quad z \in A,$$

is the degree of the map

$$z \rightarrow \frac{D_n(f_z)}{|D_n(f_z)|}$$

from $\{|z| = 1\}$ to itself.

Zjuzin [41] proved that one can take $r_n = n \cdot r_0$ for an absolute constant r_0 . Namely, the following theorem holds.

THEOREM 8.3. (Zjuzin, extending results of Gorin and Lin.) There exists a positive number r_0 with the following property. Suppose n is a prime number. If A is an annulus of conformal module $m(A) > r_0 n$, and f is a separable algebroid function on A of degree n such that the index of its discriminant is divisible by n , then f is reducible.

Petunin proved in his thesis that one can take $r_0 = 10^7$.

We will reprove now Theorem 8.3, using Theorem 1 and the following result of Penner [30].

Theorem 8.4. (Penner) Denote by h_g^s the smallest non-vanishing entropy among irreducible self-homeomorphisms of Riemann surfaces of genus g with s distinguished points ($3g - 3 + s > 0$). Then

$$h_g^s \geq \frac{\log 2}{12g - 12 + 4s}.$$

By Theorem 4 Penner's theorem implies that the smallest non-vanishing entropy among irreducible n -braids, $n \geq 3$, is bounded from below by

$$\frac{\log 2}{4(n+1) - 12} = \frac{\log 2}{4n - 8} \geq \frac{\log 2}{4} \cdot \frac{1}{n}, \quad n \geq 3.$$

We will prove now Theorem 8.3 with the constant $r_0 = \frac{2\pi}{\log 2}$.

Proof of Theorem 8.3. Suppose f is an irreducible separable algebroid function on the annulus A and

$$m(A) > \frac{\pi}{2} \frac{4}{\log 2} \cdot n = \frac{2\pi}{\log 2} \cdot n.$$

By Lemma 8.1 the conjugacy class $\hat{b}_{f,A} \in \hat{\mathcal{B}}_3$ induced by f is irreducible. By Theorem 1 the entropy $h(\hat{b}_{f,A})$ of the conjugacy class of braids $\hat{b}_{f,A} \in \hat{\mathcal{B}}_n$ induced by f is strictly smaller than $\frac{\log 2}{4} \cdot \frac{1}{n}$. By Penner's Theorem $h(\hat{b}_{f,A}) = 0$, i.e. $\mathcal{M}(\hat{b}_{f,A}) = \infty$.

This implies that $\hat{b}_{f,A}$ is the conjugacy class of a periodic braid corresponding to an n -cycle, hence it is the conjugacy class of $(\sigma_1 \sigma_2 \dots \sigma_n)^k$ for an integer k which

is not divisible by n . The isotopy class of the algebroid function $\tilde{f}(z, \zeta) = \zeta^n - z^k$, $z \in A$, induces this conjugacy class of braids. Indeed, for $z = e^{2\pi it}$, $t \in [0, 1]$, the solutions of the equation $\tilde{f}(z, \zeta) = 0$ are

$$\left\{ e^{\frac{2\pi ik}{n}t}, e^{\frac{2\pi ik}{n} + \frac{2\pi ik}{n}t}, \dots, e^{\frac{2\pi ik(n-1)}{n} + \frac{2\pi ik}{n}t} \right\}, \quad t \in [0, 1].$$

This path in \mathfrak{P}_n defines a geometric braid in the conjugacy class of $(\sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_n)^k$.

Compute the discriminant $D_n(\tilde{f}_z)$:

$$\begin{aligned} D_n(f_z) &= \prod_{0 \leq m < \ell \leq n} \left(e^{\frac{2\pi ik}{n} \cdot m + \frac{2\pi ik}{n} \cdot t} - e^{\frac{2\pi ik}{n} \cdot \ell + \frac{2\pi ik}{n} \cdot t} \right)^2 \\ (8.22) \quad &= e^{\frac{2\pi ik}{n} \cdot t \cdot n \cdot (n-1)} \cdot \prod_{0 \leq m < \ell \leq n} \left(e^{\frac{2\pi ik}{n} m} - e^{\frac{2\pi ik}{n} \ell} \right)^2. \end{aligned}$$

Hence

$$D_n(f_z) = e^{2\pi ik \cdot (n-1) \cdot t} \cdot c_n, \quad z = e^{2\pi it}, \quad t \in [0, 1],$$

where c_n is a non-zero constant depending only on n . (It is equal to the product in the last expression of (8.22).) The index of the discriminant equals $k \cdot (n-1)$ which is not divisible by n . In other words, the condition for the discriminant excludes the only possibility for a separable algebroid function on annuli of the given conformal module to be irreducible. Hence under the conditions of Theorem 8.3 the algebroid function f must be reducible.

□

CHAPTER 9

Application to elliptic fiber bundles

We will apply the concept of conformal module to fiber bundles with fibers being tori. In the following definition the fibers may be closed surfaces of any genus.

DEFINITION 9.1. *Let X be a smooth oriented manifold of dimension k (or a smooth manifold with boundary). A smooth (oriented) genus g fiber bundle \mathfrak{F}_g over X is a triple $\mathfrak{F}_g = (\mathcal{X}, \mathcal{P}, X)$ where \mathcal{X} is a smooth (oriented) manifold of dimension $k+2$ and $\mathcal{P} : \mathcal{X} \rightarrow X$ is a smooth proper submersion such that for each point $x \in X$ the fiber $\mathcal{P}^{-1}(x)$ is a closed surface of genus g .*

Denote by S a reference surface of genus g . By a theorem of Ehresmann any smooth fiber bundle of genus g over a smooth oriented manifold is locally trivial. This means that for each point $x \in X$ there is a neighbourhood $U \subset X$ of x and a surjective diffeomorphism $\varphi_U : U \times S \rightarrow \mathcal{P}^{-1}(U)$ which respects fibers:

$$\varphi_U(\{x\} \times S) = \mathcal{P}^{-1}(x) \quad \text{for each } x \in U.$$

The idea of proof of Ehresmann's Theorem is the following. Consider smooth vector fields v_j on U , which form a basis of the tangent space at each point of U . Take smooth vector fields on $\mathcal{P}^{-1}(U)$ that are mapped to v_j by the differential of \mathcal{P} . These vector fields can easily be obtained locally. To obtain globally defined vector fields on $\mathcal{P}^{-1}(U)$ one uses partitions of unity. The required diffeomorphism is obtained by composing the flows of these vector fields (in any fixed order).

Two smooth (oriented) fiber bundles of genus g over the same smooth base manifold X , $\mathfrak{F}^0 = (\mathcal{X}^0, \mathcal{P}^0, X)$, $\mathfrak{F}^1 = (\mathcal{X}^1, \mathcal{P}^1, X)$ are called (free) isotopic if there is a triple $(\mathcal{Y}, \mathcal{P}, X \times [0, 1])$ with the following properties:

- \mathcal{Y} is a smooth manifold and \mathcal{P} is a smooth proper submersion.
- For each $t \in [0, 1]$ we put $\mathcal{Y}^t = \mathcal{P}^{-1}(X \times \{t\})$. Then for each t the triple $(\mathcal{Y}^t, \mathcal{P} | \mathcal{Y}^t, X \times \{t\})$ is a smooth genus g fiber bundle.
- The bundle \mathfrak{F}^0 is isomorphic to $(\mathcal{Y}^0, \mathcal{P} | \mathcal{Y}^0, X \times \{0\})$, and the bundle \mathfrak{F}^1 is isomorphic to $(\mathcal{Y}^1, \mathcal{P} | \mathcal{Y}^1, X \times \{1\})$.

In the case when the base manifold is a Riemann surface, a holomorphic fiber bundle over X is defined as follows.

DEFINITION 9.2. *Let X be a Riemann surface. A holomorphic genus g fiber bundle over X is a triple $\mathfrak{F}_g = (\mathcal{X}, \mathcal{P}, X)$, where \mathcal{X} is a complex surface, \mathcal{P} is a holomorphic proper submersion from \mathcal{X} onto X , and each fiber is a closed Riemann surface of genus g .*

The following problem arises.

PROBLEM 9.1. *Let X be a finite Riemann surface. Is a given genus g fiber bundle on X free isotopic (through smooth bundles) to a holomorphic fiber bundle on X ?*

We will define the conformal module of isotopy classes of genus g fiber bundles over the circle which serves as obstruction for the existence of the isotopies requested in Problem 9.1.

We need the following facts.

Consider a smooth genus g fiber bundle $\mathfrak{F}_g = (\mathcal{X}, \mathcal{P}, \partial\mathbb{D})$ over the unit circle $\partial\mathbb{D}$ in the complex plane. Denote by S the fiber $\mathcal{P}^{-1}(1)$ over the point 1. There is a self-diffeomorphism φ of S so that the bundle \mathfrak{F}_g is isomorphic to the mapping torus

$$([0, 1] \times S) / ((0, \zeta) \sim (1, \varphi(\zeta))) .$$

Indeed, let v be the unit tangent vector field to $\partial\mathbb{D}$. By the argument used for the proof of Ehresmann's Theorem there is a vector field V on \mathcal{X} which projects to v , i.e. $(d\mathcal{P})(V) = v$. Cover $\partial\mathbb{D}$ by its universal covering \mathbb{R} , using the mapping $t \rightarrow e^{2\pi it}$, $t \in \mathbb{R}$. Lift the bundle \mathfrak{F}_g over $\partial\mathbb{D}$ to a bundle $\tilde{\mathfrak{F}}_g = (\tilde{\mathcal{X}}, \tilde{\mathcal{P}}, \mathbb{R})$ over \mathbb{R} , and lift the vector field V to a periodic vector field \tilde{V} on $\tilde{\mathfrak{F}}_g$. Let $\tilde{\varphi}_t(\zeta)$, $t \in \mathbb{R}$, $\zeta \in \tilde{\mathcal{P}}^{-1}(0) \cong \mathcal{P}^{-1}(1) = S$, be the solution of the differential equation

$$\frac{\partial}{\partial t} \tilde{\varphi}_t(\zeta) = \tilde{V}(\varphi_t(\zeta)), \tilde{\varphi}_t(\zeta) = \zeta \in S.$$

The time t map $\tilde{\varphi}_t(\zeta)$, $\zeta \in S$, of the vector field \tilde{V} , defines a homeomorphism from the fiber $S \cong \tilde{\mathcal{P}}^{-1}(0)$ onto the fiber $\tilde{\mathcal{P}}^{-1}(t)$. The mapping

$$S \times \mathbb{R} \ni (t, \zeta) \rightarrow (t, \varphi_t(\zeta))$$

provides a diffeomorphism from the trivial bundle to the lifted bundle $\tilde{\mathfrak{F}}_g$. Using the projection $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ we obtain a smooth family of homeomorphisms $\varphi_t : \mathcal{P}^{-1}(1) \rightarrow \mathcal{P}^{-1}(e^{2\pi it})$. Consider the time-1 map

$$\varphi_1 : \mathcal{P}^{-1}(1) \rightarrow \mathcal{P}^{-1}(e^{2\pi i}) = \mathcal{P}^{-1}(1) .$$

Notice that it depends on the bundle \mathfrak{F}_g and on the choice of the vector field V . The group \mathbb{Z} of integral numbers acts on $\mathbb{R} \times S$ by

$$\mathbb{R} \times S \ni (t, \zeta) \rightarrow (t + n, \varphi_1^{-n}(\zeta)), \quad n \in \mathbb{Z} .$$

The bundle $\mathfrak{F} = \mathfrak{F}_g$ is smoothly isomorphic to the quotient $(\mathbb{R} \times S) / \mathbb{Z}$, equivalently, \mathfrak{F}_g is smoothly isomorphic to the mapping torus

$$([0, 1] \times S) / ((0, \zeta) \sim (1, \varphi(\zeta))) .$$

The mapping $\varphi = \varphi_1^{-1}$ is called the monodromy map of the bundle \mathfrak{F} . The isotopy classes of genus g fiber bundles over the circle with fixed fiber S over 1 are in bijective correspondence to the isotopy classes of self-homeomorphisms of S . Free isotopy classes of genus g fiber bundles over the circle are in one-to-one correspondence to conjugacy classes of mapping classes of closed surfaces of genus g .

Let now X be a finite open Riemann surface. Let Q be a 1-skeleton of X . A genus g fiber bundle over Q is called smooth if for small open subsets U of X the restriction to $U \cap Q$ of the bundle extends to a smooth fiber bundle on U . Since Q is a (smooth) deformation retract of X each smooth fiber bundle over Q extends to a smooth fiber bundle over X . If the restriction to Q of two smooth fiber bundles

over X are isotopic then the fiber bundles over X are isotopic. Let $q \in Q$ be the common point of the circles of Q . Restrict the fiber bundle $(\mathcal{X}, \mathcal{P}, X)$ over \mathcal{X} to each of the circles of Q . Consider the monodromy map at the base point q over each of the circles. We obtain the following known theorem (see, e.g. [11]).

THEOREM 9.1. *Let X be a smooth open surface with finitely generated fundamental group. The set of isotopy classes of smooth oriented genus g fiber bundles on X is in one-to-one correspondence to the set of conjugacy classes of homomorphisms from the fundamental group $\pi_1(X)$ into the modular group $\text{Mod}_g \stackrel{\text{def}}{=} \text{Mod}(g, 0)$ of closed Riemann surfaces of genus g .*

We are now ready to define the conformal module of isotopy classes of genus g fiber bundles. Consider a smooth oriented genus g fiber bundle $\mathfrak{F} = \mathfrak{F}_g = (\mathcal{X}, \mathcal{P}, \partial\mathbb{D})$ over the circle. Denote by $\widehat{\mathfrak{F}}$ the free isotopy class of fiber bundles over $\partial\mathbb{D}$ that contains \mathfrak{F} . Let $A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$, $r < 1 < R$, be an annulus containing the unit circle. A fiber bundle on $A_{r,R}$ is said to represent $\widehat{\mathfrak{F}}$ if its restriction to the unit circle $\partial\mathbb{D}$ is an element of $\widehat{\mathfrak{F}}$.

DEFINITION 9.3. (The conformal module of isotopy classes of genus g fiber bundles.) *Let $\widehat{\mathfrak{F}} = \widehat{\mathfrak{F}}_g$ be the free isotopy class of an oriented genus g fiber bundle over the circle $\partial\mathbb{D}$. Its conformal module is defined as*

$$\mathcal{M}(\widehat{\mathfrak{F}}) =$$

$\sup\{m(A_{r,R}) : \text{there exists a holomorphic fiber bundle on } A_{r,R} \text{ that represents } \widehat{\mathfrak{F}}\}.$

Let $\mathfrak{F} \in \widehat{\mathfrak{F}}$ be a fiber bundle over $\partial\mathbb{D}$ and let $\varphi_{\mathfrak{F}}$ be the monodromy map of \mathfrak{F} . Denote by $\mathfrak{m}_{\mathfrak{F}}$ the mapping class of $\varphi_{\mathfrak{F}}$ on the fiber $\mathcal{P}^{-1}(1)$ over 1 of the bundle \mathfrak{F} . Let $\widehat{\mathfrak{m}}_{\mathfrak{F}}$ be the conjugacy class of $\mathfrak{m}_{\mathfrak{F}}$. Notice that $\widehat{\mathfrak{m}}_{\mathfrak{F}}$ depends only on $\widehat{\mathfrak{F}}$. Hence, the conformal module $\mathcal{M}(\widehat{\mathfrak{F}})$ is an invariant of the conjugacy class $\widehat{\mathfrak{m}}_{\mathfrak{F}}$. We will write also

$$(9.1) \quad \mathcal{M}(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = \mathcal{M}(\widehat{\mathfrak{F}}).$$

In the following we will consider oriented genus 1 fiber bundles. They are called elliptic fiber bundles.

Let \mathfrak{F} be an elliptic fiber bundle over the circle $\partial\mathbb{D}$. We may give the fiber over $1 \in \partial\mathbb{D}$ the complex structure of the standard torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Let φ be the monodromy map of the bundle. Since translations on a torus are isotopic to the identity we may assume after a free isotopy of the bundle that φ fixes a point. Consider the lift $\tilde{\varphi}$ of φ to the universal covering \mathbb{C} of the torus. We may assume that $\tilde{\varphi}$ fixes the point $0 \in \mathbb{C}$. After a further isotopy we may assume that $\tilde{\varphi}$ is an orientation preserving real linear self-map of \mathbb{C} which maps the integer lattice $\mathbb{Z} + i\mathbb{Z}$ onto itself. In other words, $\tilde{\varphi}$ corresponds to a 2×2 matrix A with integral entries and determinant 1, $\varphi(x + iy) = A \begin{pmatrix} x \\ y \end{pmatrix}$, $A \in \text{SL}_2(\mathbb{Z})$. The bundle \mathfrak{F} is free isotopic to the quotient bundle

$$(9.2) \quad (\mathbb{R} \times \mathbb{C}) / ((t, \zeta) \sim (t, \zeta + n + im) \sim (t + 1, \tilde{\varphi}(\zeta)))$$

for all integers n and m .

We may represent the isotopy class $\widehat{\mathfrak{F}}$ by a bundle with explicitly given fibers equipped with complex structure. This is done as follows. Let $\tilde{\varphi}_t$, $t \in [0, 1]$, be a

smooth family of real linear self-maps of \mathbb{C} with $\tilde{\varphi}_t = \text{id}$ for t close to 1 and $\tilde{\varphi}_t = \tilde{\varphi}$ for t close to 0. For each $t \in [0, 1]$ we consider the lattice $\tilde{\varphi}_t(\mathbb{Z} + i\mathbb{Z})$. Notice that $\tilde{\varphi}_1(\mathbb{Z} + i\mathbb{Z}) = \mathbb{Z} + i\mathbb{Z}$, and $\tilde{\varphi}_1(\mathbb{Z} + i\mathbb{Z}) = \tilde{\varphi}_0(\mathbb{Z} + i\mathbb{Z})$. Hence, the lattices depend only on $e^{2\pi it}$, $t \in [0, 1]$. We denote the lattice $\tilde{\varphi}_t(\mathbb{Z} + i\mathbb{Z})$ by $\Lambda(e^{2\pi it})$, $t \in [0, 1]$.

We obtain a mapping

$$(9.3) \quad \partial\mathbb{D} \ni e^{2\pi it} \rightarrow \Lambda(e^{2\pi it}),$$

which is smooth in the sense that for each $e^{2\pi it_0} \in \partial\mathbb{D}$ there is an arc $\gamma \subset \partial\mathbb{D}$ around $e^{2\pi it_0}$ such that for $e^{2\pi it} \in \gamma$ the lattices can be written as

$$(9.4) \quad \Lambda(e^{2\pi it}) = a(e^{2\pi it})\mathbb{Z} + b(e^{2\pi it})\mathbb{Z}$$

for smooth complex valued functions a and b such that for each t the numbers $a(e^{2\pi it})$ and $b(e^{2\pi it})$ are linearly independent over t . Notice that the quotient of \mathbb{C} by a lattice determines the lattice. We arrive at the following statement.

One can associate to any smooth isotopy class $\hat{\mathfrak{F}}$ a smooth mapping

$$(9.5) \quad \partial\mathbb{D} \ni e^{2\pi it} \rightarrow \mathbb{C}/\Lambda(e^{2\pi it}), \quad \Lambda(1) = \mathbb{Z} + i\mathbb{Z}.$$

Here we call a mapping of the form (9.5) smooth if the respective mapping of the form (9.4) is smooth.

Vice versa, a smooth mapping of the form (9.5) defines a self-homeomorphism of the torus $\mathbb{C}/\Lambda(1)$ and, hence, an elliptic fiber bundle with all fibers equipped with complex structure.

Indeed, if on an arc we have $\tilde{\Lambda}(e^{it}) = \Lambda(e^{it})$, with $\tilde{\Lambda}(e^{it}) = \tilde{a}(e^{it})\mathbb{Z} + \tilde{b}(e^{it})\mathbb{Z}$ for other smooth functions \tilde{a}, \tilde{b} , then there is a matrix $B_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ such that

$$\tilde{a}(e^{it}) = \alpha_1 a(e^{it}) + \gamma_1 b(e^{it}), \quad \tilde{b}(e^{it}) = \beta_1 a(e^{it}) + \delta_1 b(e^{it})$$

for all points e^{it} in the arc. Indeed, $\Lambda(e^{it})$ and $\tilde{\Lambda}(e^{it})$ are equal lattices and $(a(e^{it}), b(e^{it}))$ and $(\tilde{a}(e^{it}), \tilde{b}(e^{it}))$ are pairs of generators of the lattice. Put $A(e^{it}) = \begin{pmatrix} \text{Re } a(e^{it}) & \text{Re } b(e^{it}) \\ \text{Im } a(e^{it}) & \text{Im } b(e^{it}) \end{pmatrix}$ and $\tilde{A}(e^{it}) = \begin{pmatrix} \text{Re } \tilde{a}(e^{it}) & \text{Re } \tilde{b}(e^{it}) \\ \text{Im } \tilde{a}(e^{it}) & \text{Im } \tilde{b}(e^{it}) \end{pmatrix}$. Then the real linear self-maps of \mathbb{C} defined by $A(e^{it})$ and $\tilde{A}(e^{it})$ map the standard lattice $\mathbb{Z} + i\mathbb{Z}$ onto $\Lambda(e^{it}) = \tilde{\Lambda}(e^{it})$. Then $A^{-1}(e^{it}) \circ \tilde{A}(e^{it})$ maps the standard lattice to itself, hence, it is in $\text{SL}_2(\mathbb{Z})$. Since $A(e^{it})$ and $\tilde{A}(e^{it})$ depend continuously on e^{it} , the matrices $A^{-1}(e^{it}) \circ \tilde{A}(e^{it})$ are equal to a constant matrix $B_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$.

Cover the circle by a suitable collection of open arcs. We obtain that there are smooth functions \tilde{a} and \tilde{b} on $(-\varepsilon, 1 + \varepsilon)$ so that $\Lambda(e^{it}) = \tilde{a}(t)\mathbb{Z} + \tilde{b}(t)\mathbb{Z}$, $t \in (-\varepsilon, 1 + \varepsilon)$, and

$$\tilde{a}(t + 2\pi) = \alpha a(t) + \gamma b(t), \quad \tilde{b}(t + 2\pi) = \beta a(t) + \delta b(t).$$

Here B is a matrix in $\text{SL}_2(\mathbb{Z})$.

The real linear map corresponding to B maps $(\tilde{a}(0), \tilde{b}(0))$ to $(\tilde{a}(2\pi), \tilde{b}(2\pi))$. In particular, it maps the lattice $\Lambda(1)$ onto itself, and, hence, descends to a self-homeomorphism of the fiber $\mathbb{C}/\Lambda(1)$ associated to (9.5).

Proceed in a similar way with isotopy classes $\hat{\mathfrak{F}}$ of bundles over finite Riemann surfaces. Let X be a finite open Riemann surface and let $\mathfrak{F} \in \hat{\mathfrak{F}}$ be a smooth

oriented elliptic fiber bundle over X . Restrict \mathfrak{F} to a skeleton Q of X . Let $z_0 \in Q$ be the common point of the circles of Q . After an isotopy we may assume that the monodromy maps at z_0 fix a point ζ_0 in the fiber over z_0 . Proceed with each circle of Q as above we obtain a mapping

$$(9.6) \quad Q \ni z \rightarrow \mathbb{C}/\Lambda(z).$$

Using a smooth deformation retraction of X onto Q we obtain a mapping

$$(9.7) \quad X \ni z \rightarrow \mathbb{C}/\Lambda(z).$$

The mapping is smooth in the sense that the mapping $X \ni z \rightarrow \Lambda(z)$ is smooth. (Recall that the quotient \mathbb{C}/Λ determines the lattice Λ .) The mapping $X \ni z \rightarrow \Lambda(z)$ is called smooth if for each $z_0 \in X$ there is a neighbourhood $U(z_0) \subset X$ such that the lattices $\Lambda(z)$ can be written as

$$\Lambda(z) = a(z)\mathbb{Z} + b(z)\mathbb{Z}.$$

for smooth functions a and b on $U(z_0)$. If on $U(z_0)$ we have $\tilde{\Lambda}(z) = \Lambda(z)$, with $\tilde{\Lambda}(z) = \tilde{a}(z)\mathbb{Z} + \tilde{b}(z)\mathbb{Z}$ for other smooth functions \tilde{a}, \tilde{b} , then there is a matrix $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$(9.8) \quad \tilde{a}(z) = \alpha a(z) + \gamma b(z), \quad \tilde{b}(z) = \beta a(z) + \delta b(z).$$

The mapping (9.7) defines a bundle, which is locally given as quotient of $U(z_0) \times \mathbb{C}$ by the equivalence relation

$$(9.9) \quad U(z_0) \times \mathbb{C} \ni (z, \zeta) \sim (z, \zeta + a(z)n + b(z)m)$$

for all integers n and m . Here z_0 is an arbitrary point in X and $U(z_0)$ is a suitably small neighbourhood of z_0 , and a and b are smooth functions. The transition functions between charts of the form determined by (9.8) are smooth. This bundle represents $\hat{\mathfrak{F}}$. We will also say that the mapping (9.7) represents $\hat{\mathfrak{F}}$.

Notice that from the very beginning we have achieved by isotopy that the bundle admits a smooth section $X \ni z \rightarrow s(z) \in \mathbb{C}/\Lambda(z)$.

The bundle given by (9.7) can be considered as differentiable family of complex manifolds (here complex one-dimensional tori) in the sense of the theory of deformation of complex manifolds ([23]). Definition 9.2 describes what is called a complex analytic family of complex manifolds in deformation theory. Problem 9.1 can be reformulated as follows.

Which differentiable families of closed Riemann surfaces of genus g over a finite open Riemann surface X can be deformed to a complex analytic family?

For a smooth elliptic fiber bundle with complex fibers just the complex structure of all fibers determines the isotopy class of monodromy maps up to composition with a periodic map. Indeed, note first that for two lattices Λ_1 and Λ_2 the quotients \mathbb{C}/Λ_1 and \mathbb{C}/Λ_2 are conformally equivalent if and only if $\Lambda_2 = \alpha \cdot \Lambda_1$ for a complex number $\alpha \neq 0$. Indeed, if the quotients are conformally equivalent then there is a conformal map from \mathbb{C}/Λ_1 onto \mathbb{C}/Λ_2 which lifts to a complex affine map

$$\zeta \rightarrow \beta + \alpha \zeta, \quad \zeta \in \mathbb{C}.$$

Hence, there is another conformal map which lifts to a complex linear self-map $\zeta \rightarrow \alpha \zeta$ of \mathbb{C} . Thus, $\Lambda_2 = \alpha \Lambda_1$.

Vice versa if $\Lambda_2 = \alpha \Lambda_1$ then the two quotients are conformally equivalent.

The linear map $\zeta \rightarrow \alpha \zeta$, $\zeta \in \mathbb{C}$, maps the set of elements of Λ_1 of smallest absolute value onto the set of elements of Λ_2 of smallest absolute value. Hence α is defined up to multiplication with a fourth root of unity, and unless Λ_1 is a multiple of $\mathbb{Z} + i\mathbb{Z}$ it is defined up to sign.

Take now two smooth bundles with fiberwise equal complex structures. After isotoping each of the bundles we obtain two smooth bundles of the form (9.7) with lattices differing by multiplication with non-zero complex numbers $\alpha(z)$ which depend smoothly on z . Hence the bundles are isotopic.

Let now $\mathfrak{F} = (\mathcal{X}, \mathcal{P}, X)$ be a *holomorphic* elliptic fiber bundle over a finite Riemann surface X . Let $\Delta \subset X$ be any small enough topological disc. A smooth trivialization of the bundle over Δ defines for each $z \in \Delta$ a surjective orientation preserving homeomorphism $\omega_z : \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \rightarrow \mathcal{P}^{-1}(z)$ onto the fiber $\mathcal{P}^{-1}(z)$ of the bundle over z . Notice, that each fiber is a complex manifold and is conformally equivalent to a quotient $\mathbb{C}/\Lambda(z)$ for a lattice $\Lambda(z) = a(z)\mathbb{Z} + b(z)\mathbb{Z}$ with $a(z) = \omega_z(1)$ and $b(z) = \omega_z(i)$ linearly independent over \mathbb{R} . Let $[w_z]$ be the Teichmüller class of ω_z in $\mathcal{T}(\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}))$. The Teichmüller class is determined by the complex structure of the fiber and by the smooth trivialization. Notice that $\text{Im} \frac{a(z)}{b(z)} > 0$ and the Teichmüller class $[w_z]$ can be identified with the number $\tau(z) = \frac{a(z)}{b(z)} \in \mathbb{C}_+$. The following lemma holds.

LEMMA 9.1. *Let \mathfrak{F} be a holomorphic elliptic fiber bundle over a Riemann surface X . For each small enough disc $\Delta \subset X$ the induced map $z \rightarrow \tau(z)$, $z \in \Delta$, is holomorphic.*

For completeness we give the proof. The key ingredient is a lemma of Kodaira which we formulate now.

Let \mathcal{X} and X be complex manifolds and let $\mathcal{P} : \mathcal{X} \rightarrow X$ be a proper holomorphic submersion such that the fibers $\mathcal{X}_z = \mathcal{P}^{-1}(z)$, $z \in X$, are compact complex manifolds of complex dimension n . For each $z \in X$ we denote by Θ_z the sheaf of germs of holomorphic tangent vector fields of the complex manifold \mathcal{X}_z . Denote by $H^0(\mathcal{X}_z, \Theta_z)$ the space of global sections of the sheaf.

LEMMA 9.2. (Kodaira, [23], Lemma 4.1, p. 204.) *If the dimension of $H^0(\mathcal{X}_z, \Theta_z) = d$ is independent of z then for any small enough (topological) ball Δ in X there is for each $z \in \Delta$ a basis $(v_1(z), \dots, v_d(z))$ of $H^0(\mathcal{X}_z, \Theta_z)$ such that $(v_1(z), \dots, v_d(z))$ depends holomorphically on $z \in \Delta$.*

Let m be the dimension of X . The condition that $v_j(z)$, $j = 1, \dots, d$, depends holomorphically on z , means the following. Let $U_\alpha \subset \mathcal{X}$ be a small open subset of \mathcal{X} on which there are local holomorphic coordinates $(\zeta_1^\alpha, \dots, \zeta_n^\alpha, z_1, \dots, z_m)$, where $z = (z_1, \dots, z_m) \in \Delta$ and for fixed $z = (z_1, \dots, z_m)$ the $\zeta^\alpha = (\zeta_1^\alpha, \dots, \zeta_n^\alpha)$ are local holomorphic coordinates on the fiber over z . In these local coordinates the vector field v_j , $j = 1, \dots, n$, can be written as

$$(9.10) \quad v_j(z) = \sum_{k=1}^n v_{jk}^\alpha(\zeta^\alpha, z) \frac{\partial}{\partial \zeta_k^\alpha},$$

where the v_{jk}^α are holomorphic in (ζ^α, z) . The condition does not depend on the choice of local coordinates with the described properties.

Proof of Lemma 9.1. In the situation of Lemma 9.1 the base X has complex dimension one and the fibers are complex tori of complex dimension one. The space of holomorphic sections $H^0(\mathcal{X}_z, \Theta_z)$ of holomorphic tangent vector fields of the fiber \mathcal{X}_z has complex dimension one. Indeed, the dimension of the space of holomorphic 1-forms on a closed Riemann surface of genus g equals g (see, e.g. [10], p. 73). Thus the dimension of the space of holomorphic 1-forms on a closed torus equals one and therefore the space of sections of the holomorphic tangent bundle to the torus has complex dimension one. We see that for each $z \in \Delta$ the space $H^0(\mathcal{X}_z, \Theta_z)$ is generated by a single holomorphic tangent vector field $v(z)$ on \mathcal{X}_z . By Kodaira's Lemma $v(z)$ may be chosen to depend holomorphically on $z \in \Delta$ for a small disc Δ in X . We obtain a holomorphic vector field v on $\mathcal{X}_\Delta = \mathcal{P}^{-1}(\Delta)$ whose restriction to each fiber \mathcal{X}_z equals $v(z)$.

Consider the (holomorphic) universal covering $\tilde{\mathcal{X}}_\Delta$ of \mathcal{X}_Δ .

LEMMA 9.3. $\tilde{\mathcal{X}}_\Delta$ is a trivial holomorphic fiber bundle over Δ with fiber \mathbb{C} .

Proof of Lemma 9.3. \mathcal{X}_Δ is smoothly equivalent to $\Delta \times T_0$, where $T_0 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is the standard torus. The smooth universal covering of $\Delta \times T_0$ is $\Delta \times \mathbb{R}^2$. We get the following commutative diagram:

$$\begin{array}{ccc} \Delta \times \mathbb{R}^2 & & \\ \downarrow \tilde{p} & \searrow p & \\ \Delta \times T_0 & \xrightarrow{F} \mathcal{X}_\Delta & \xrightarrow{\mathcal{P}} \Delta, \end{array}$$

where \mathcal{P} is holomorphic, F is smooth, \tilde{p} is the smooth universal covering map from the universal covering $\Delta \times \mathbb{R}^2$ of $\Delta \times T_0$ onto $\Delta \times T_0$ and the smooth map p is defined as composition $F \circ \tilde{p}$. Notice that $(\mathcal{P} \circ p)^{-1}(z) = z \times \mathbb{R}^2$ for $z \in \Delta$.

Equip $\Delta \times \mathbb{R}^2$ with the complex structure for which p is holomorphic. The obtained complex manifold is (holomorphically isomorphic to) $\tilde{\mathcal{X}}_\Delta$. Then $(\mathcal{P} \circ p)^{-1}(z)$ is a complex submanifold of $\tilde{\mathcal{X}}_\Delta$ and $p|_{(\mathcal{P} \circ p)^{-1}(z)}$ is a holomorphic covering. Since the covering manifold $(\mathcal{P} \circ p)^{-1}(z)$ is diffeomorphic to \mathbb{R}^2 and, hence, simply connected, and the covered manifold $\mathcal{P}^{-1}(z)$ is a torus, the set $(\mathcal{P} \circ p)^{-1}(z)$ is conformally equivalent to \mathbb{C} . Hence, the triple $(\tilde{\mathcal{X}}_\Delta, \mathcal{P} \circ p, \Delta)$ is a holomorphic fiber bundle with fiber \mathbb{C} .

Take a holomorphic section $s(z)$ of the bundle $\tilde{\mathcal{X}}_\Delta \xrightarrow{\mathcal{P} \circ p} \Delta$. It exists after, perhaps, shrinking Δ . Let \tilde{v} be a lift of the vector field v on \mathcal{X}_Δ to the universal covering $\tilde{\mathcal{X}}_\Delta$. Define a mapping $\mathcal{G} : \Delta \times \mathbb{C} \rightarrow \tilde{\mathcal{X}}_\Delta$ by

$$(9.11) \quad \mathcal{G}(z, \zeta) = \gamma_{s(z)}(\zeta), \quad (z, \zeta) \in \Delta \times \mathbb{C},$$

where for each z the mapping $\gamma_{s(z)}$ is the solution of the holomorphic differential equation

$$(9.12) \quad \gamma'_{s(z)}(\zeta) = \tilde{v}(\gamma_{s(z)}(\zeta)), \quad \gamma_{s(z)}(0) = s(z) \in (\mathcal{P} \circ p)^{-1}(z), \quad \zeta \in \mathbb{C}.$$

Since \tilde{v} is tangential to the fibers we have the inclusion

$$(9.13) \quad \gamma_{s(z)}(\mathbb{C}) \subset (\mathcal{P} \circ p)^{-1}(z) \cong \mathbb{C} \quad \text{for each } z \in \Delta.$$

For each z the solution exists for all $\zeta \in \mathbb{C}$ since the restriction of \tilde{v} to the fiber over z is the lift of a vector field on a closed torus. The mapping

$$(9.14) \quad (z, \zeta) \rightarrow \gamma_{s(z)}(\zeta), \quad z \in \Delta, \quad \zeta \in \mathbb{C},$$

is a holomorphic diffeomorphism onto its image. For each z it maps \mathbb{C} one-to-one into the fiber $(\mathcal{P} \circ p)^{-1}(z) \cong \mathbb{C}$, hence it maps \mathbb{C} *onto* $(\mathcal{P} \circ p)^{-1}(z)$. Hence \mathcal{G} defines a holomorphic isomorphism of the trivial bundle $\Delta \times \mathbb{C} \rightarrow \Delta$ onto the bundle $\tilde{\mathcal{X}}_\Delta \xrightarrow{\mathcal{P} \circ p} \Delta$. Lemma 9.3 is proved. \square

Consider the covering transformations of the covering $\tilde{\mathcal{X}}_\Delta \rightarrow \mathcal{X}_\Delta$. In terms of the isomorphic bundle $\Delta \times \mathbb{C} \rightarrow \Delta$ the restrictions of the covering transformations to each fiber $\{z\} \times \mathbb{C}$, are translations, hence, the covering transformations have the form

$$(9.15) \quad \psi_{n,m}(z, \zeta) \rightarrow \zeta + n a(z) + m b(z), \quad (z, \zeta) \in \Delta \times \mathbb{C},$$

for integral numbers n and m . Here $a(z)$ and $b(z)$ are complex numbers which are linearly independent over \mathbb{R} and depend on z . Since the covering maps $\psi_{1,0}$ and $\psi_{0,1}$ are holomorphic, the numbers $a(z)$ and $b(z)$ depend holomorphically on $z \in \Delta$. Hence $\tau(z) = \frac{a(z)}{b(z)}$ depends holomorphically on $z \in \Delta$. Lemma 9.1 is proved. \square

We will now describe how to use Lemma 9.1 to represent the isotopy class of a *holomorphic* elliptic fiber bundle over a Riemann surface X by a *holomorphic* map of the form (9.7). For this we choose a connected simply connected open subset of the universal covering of X such that each point of X is covered either once or twice. We will then lift the bundle to this set.

The choice can be made as follows. Let X be a connected finite open Riemann surface and let $q \in X$ be a base point. Let X^c be the connected closed Riemann surface obtained from X by adding topological discs or points. Consider the universal covering $\tilde{X}^c \xrightarrow{p} X^c$. Fix a point \tilde{q} over q , i.e. $p(\tilde{q}) = q$. Denote the closure on \tilde{X}^c of a fundamental domain for X^c by $F(X^c)$. We may assume that the interior of $F(X^c)$ contains \tilde{q} . The intersection of $F(X^c)$ with the preimage $p^{-1}(X)$ is denoted by $F(X)$. Take a small neighbourhood on $p^{-1}(X)$ of $F(X)$ and denote it by $UF(X)$. We may assume that $UF(X)$ is close enough to $F(X)$, so that $UF(X)$ covers points in X at most twice.

Take a smooth elliptic fiber bundle $(\mathcal{X}, \mathcal{P}, X)$ over X , and lift it to $UF(X)$. The lifted bundle is smoothly trivial over $UF(X)$. Take a skeleton $Q \subset X \subset X^c$ of X such that q is the common point of the circles constituting the skeleton, and consider the intersection $F(Q)$ of the preimage $\mathcal{P}^{-1}(Q)$ with $UF(X)$. We may assume that $UF(X)$ is close enough to $F(X)$ so that $F(Q) = p^{-1}(Q) \cap UF(X)$ is a connected graph with a single multiple vertex \tilde{q} and each circle of Q is covered by two edges of the graph. A trivialization of the restriction of the lifted bundle to $F(Q)$ is obtained by applying to each circle of Q the arguments used for bundles over the circle. A trivialization of the bundle over $UF(X)$ is obtained by using that $F(Q)$ is a deformation retract of $UF(X)$.

Consider now a holomorphic elliptic fiber bundle \mathfrak{F} over the open Riemann surface X . Denote by $\tilde{\mathfrak{F}} = (\tilde{UF}(X), \tilde{\mathcal{P}}, UF(X))$ a lift of the bundle to $UF(X)$. The bundle \mathfrak{F} is obtained from $\tilde{\mathfrak{F}}$ by taking the quotient with respect to the monodromy maps. More detailed, let $\varphi_{\tilde{z}} : \tilde{\mathcal{P}}^{-1}(\tilde{q}) \rightarrow \tilde{\mathcal{P}}^{-1}(\tilde{z})$ be the smooth family of diffeomorphisms which trivialize the bundle $\tilde{\mathfrak{F}}$.

Consider the projection $UF(X) \rightarrow X$, and the (open) set of points of X covered twice by $UF(X)$. We may assume that the connected components U_j of this set are in bijective correspondence to the circles of Q and each circle intersects exactly the component U_j associated to it. The respective pairs of subsets of $UF(X)$ that cover U_j are denoted by \tilde{U}_j^- and \tilde{U}_j^+ (see figure 9.1).

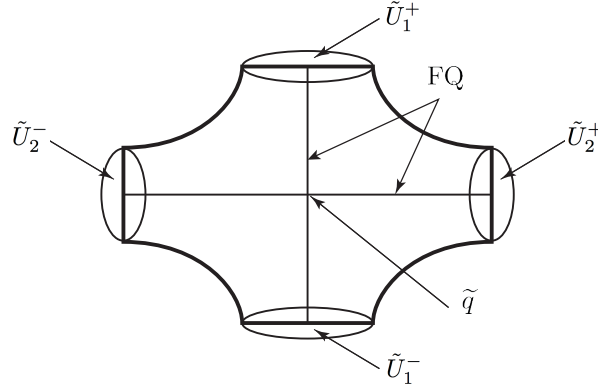


FIGURE 9.1

The complex structure of the fibers $\tilde{\mathcal{P}}^{-1}(\tilde{z})$, $\tilde{z} \in UF(X)$, together with the smooth trivialization of the bundle $\tilde{\mathfrak{F}}$ define a map $UF(X) \ni \tilde{z} \rightarrow \tau(\tilde{z})$ into the Teichmüller space $\mathcal{T}(\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}))$ which is holomorphic by Lemma 9.1.

Take an arbitrary point in U_j and denote it by z_j . The points over z_j are $\tilde{z}_j^- \in \tilde{U}_j^-$ and $\tilde{z}_j^+ \in \tilde{U}_j^+$. The fibers $\tilde{\mathcal{P}}^{-1}(\tilde{z}_j^-)$ and $\tilde{\mathcal{P}}^{-1}(\tilde{z}_j^+)$ are conformally equivalent. Hence $\tau(\tilde{z}_j^+)$ is related to $\tau(\tilde{z}_j^-)$ by a modular transformation φ_j^* on the Teichmüller space $\mathcal{T}(\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}))$, namely, we have $\tau(\tilde{z}_j^+) = w_{\tau(\tilde{z}_j^+)} \circ \varphi_j^* \circ w_{\tau(\tilde{z}_j^+)}^{-1}$.

The bundle $\tilde{\mathfrak{F}}$ over $UF(X)$ is isotopic to any bundle of the form

$$(9.16) \quad UF(X) \ni \tilde{z} \rightarrow \mathbb{C}/\alpha(\tilde{z}) \ (\mathbb{Z} + \tau(\tilde{z})\mathbb{Z})$$

where α is any nowhere vanishing holomorphic function on $UF(X)$.

Since for each j and $z_j \in U_j$ the fibers $\tilde{\mathcal{P}}^{-1}(\tilde{z}_j^-) \cong \mathbb{C}/(\mathbb{Z} + \tau(\tilde{z}_j^-)\mathbb{Z})$ and $\tilde{\mathcal{P}}^{-1}(\tilde{z}_j^+) \cong \mathbb{C}/(\mathbb{Z} + \tau(\tilde{z}_j^+)\mathbb{Z})$ are conformally equivalent, there is for each j a family of non-zero complex numbers $\alpha_j(z_j)$ which depend holomorphically on $z_j \in U_j$ such that

$$(9.17) \quad \mathbb{Z} + \tau(\tilde{z}_j^+)\mathbb{Z} = \alpha_j(z_j) (\mathbb{Z} + \tau(\tilde{z}_j^-)\mathbb{Z}), \quad z_j \in U_j.$$

For generic τ the numbers α for which $\mathbb{Z} + \tau\mathbb{Z} = \alpha(\mathbb{Z} + \tau\mathbb{Z})$ are defined up to sign. Since the function α_j is continuous it is defined up to sign provided the function $\tau(\tilde{z})$ is not constant.

We want to choose the holomorphic function a so that the lattices $a(\tilde{z}) (\mathbb{Z} + \tau(\tilde{z})\mathbb{Z})$ coincide for the values \tilde{z}_j^- and \tilde{z}_j^+ over the same point $z_j \in U_j$. This condition is equivalent to the relation

$$(9.18) \quad a(\tilde{z}_j^+) = \alpha_j(z_j) \cdot a(\tilde{z}_j^-), \quad \tilde{z}_j^- \in \tilde{U}_j^-.$$

The existence of such a holomorphic function a on $UF(X)$ is a consequence of the triviality of the second cohomology $H^2(X, \mathbb{Z})$ with integral coefficients for the open Riemann surface X ([13], [14], Proposition V.1.8). The construction of the function a actually leads to solving a $\bar{\partial}$ -problem on X as in the case of problem (*) in chapter 5.

For each j we make a choice of α_j and consider the obtained function a on $UF(X)$. We obtain a mapping

$$UF(X) \ni \tilde{z} \rightarrow \tilde{\Lambda}(\tilde{z}) = \alpha(\tilde{z}) (\mathbb{Z} + \tau(\tilde{z})\mathbb{Z}) = \mathbf{a}(\tilde{z})\mathbb{Z} + \mathbf{b}(\tilde{z})\mathbb{Z},$$

such that $\tilde{\Lambda}(\tilde{z}_j^-) = \tilde{\Lambda}(\tilde{z}_j^+)$. Hence $\tilde{\Lambda}(\tilde{z})$ descends to a holomorphic family of lattices $\Lambda(z)$, $z \in X$. The family defines isotopy classes \mathbf{m}_j , $j = 1, 2$, of monodromy maps at a fixed point $z_0 \in X$.

The isotopy class \mathbf{m}_j is defined by φ_j^* up to a conformal self-map of the fibers. Looking at a generic point and using the continuous dependence of τ on the point \tilde{z} we see that either \mathbf{m}_j is conjugate to the respective monodromy map of the bundle \mathfrak{F} or to the composition of the latter monodromy map with the involution ι . The latter case corresponds to changing α_j to $-\alpha_j$. Change α to the product of α with the solution of the problem (9.18) with $\alpha_j = (-1)^{\ell_j}$, $j = 1, 2$. We obtain a bundle which is isotopic to the bundle \mathfrak{F} and has the requested form.

We obtained the following statement:

The isotopy class $\widehat{\mathfrak{F}}$ of a holomorphic elliptic fiber bundle over a connected finite open Riemann surface can be represented by a holomorphic map

$$(9.19) \quad X \ni z \rightarrow \mathbb{C}/\Lambda(z).$$

We want to reduce the study of elliptic fiber bundles to the study of quasipolynomials of degree 3. For this purpose it will be convenient to represent closed tori as double branched coverings.

For a lattice Λ the torus \mathbb{C}/Λ is conformally equivalent to a double branched cover of \mathbb{P}^1 with four branch points. The set of branch points is not uniquely determined by the conformal equivalence class of the torus. We make a choice using a holomorphic embedding of the punctured torus $\mathbb{C} \setminus \Lambda / \Lambda$ into \mathbb{C}^2 as follows.

Suppose $\Lambda = \alpha(\mathbb{Z} + \tau\mathbb{Z})$ ($\tau \in \mathbb{C}_+$, $\alpha \in \mathbb{C} \setminus \{0\}$). Consider the mapping

$$(9.20) \quad \mathbb{C} \setminus \Lambda \ni \zeta \rightarrow \left(\alpha^{-2} \wp_\tau \left(\frac{\zeta}{\alpha} \right), \alpha^{-3} \wp'_\tau \left(\frac{\zeta}{\alpha} \right) \right) \in \mathbb{C}^2.$$

Here \wp_τ is the Weierstraß \wp -function,

$$(9.21) \quad \wp_\tau(\zeta) = \frac{1}{\zeta^2} + \sum_{\substack{(n,m) \in \mathbb{Z}^2 \\ (n,m) \neq (0,0)}} \left(\frac{1}{(\zeta - n - m\tau)^2} - \frac{1}{(n + m\tau)^2} \right), \quad \zeta \in \mathbb{C} \setminus (\mathbb{Z} + \tau\mathbb{Z}).$$

The function \wp_τ is meromorphic on \mathbb{C} , has poles of second order at points of $\mathbb{Z} + \tau\mathbb{Z}$, and is holomorphic on $\mathbb{C} \setminus (\mathbb{Z} + \tau\mathbb{Z})$. It has periods 1 and τ and principal part $\zeta \rightarrow \frac{1}{\zeta^2}$ at 0. The function $\zeta \rightarrow \alpha^{-2} \wp_\tau \left(\frac{\zeta}{\alpha} \right)$ is holomorphic on $\mathbb{C} \setminus \Lambda$, its periods are α and $\alpha\tau$, and its principal part at 0 is $\zeta \rightarrow \frac{1}{\zeta^2}$. The mapping (9.20) defines

an embedding of the punctured torus $\mathbb{C} \backslash \Lambda / \Lambda$ into \mathbb{C}^2 . We identify the punctured torus with the image of (9.20).

The function \wp_τ satisfies the differential equation

$$(9.22) \quad (\wp'_\tau)^2(\zeta) = 4(\wp_\tau(\zeta) - e_1(\tau))(\wp_\tau(\zeta) - e_2(\tau))(\wp_\tau(\zeta) - e_3(\tau)),$$

where

$$(9.23) \quad e_1(\tau) = \wp_\tau\left(\frac{1}{2}\right), \quad e_2(\tau) = \wp_\tau\left(\frac{\tau}{2}\right), \quad e_3(\tau) = \wp_\tau\left(\frac{1+\tau}{2}\right)$$

are the values of \wp_τ at points which are contained in the lattice $\frac{1}{2}\mathbb{Z} + \frac{\tau}{2}\mathbb{Z}$ but are not contained in the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Equation (9.22) defines a double branched cover of \mathbb{C}

$$(9.24) \quad w^2 = 4(z - e_1(\tau))(z - e_2(\tau))(z - e_3(\tau))$$

with branch locus $\{e_1(\tau), e_2(\tau), e_3(\tau)\}$.

If we put $w = \alpha^{-3} \wp'_\tau\left(\frac{\zeta}{\alpha}\right)$, $z = \alpha^{-2} \wp_\tau\left(\frac{\zeta}{\alpha}\right)$, we obtain the double branched cover of \mathbb{C}

$$(9.25) \quad w^2 = (z - \alpha^{-2} e_1(\tau))(z - \alpha^{-2} e_2(\tau))(z - \alpha^{-2} e_3(\tau)).$$

The branch locus $\{\alpha^{-2} e_1(\tau), \alpha^{-2} e_2(\tau), \alpha^{-2} e_3(\tau)\}$ is the set of values of $\alpha^{-2} \wp_\tau\left(\frac{\zeta}{\alpha}\right)$ at the points which are contained in the lattice $\frac{1}{2}\Lambda = \frac{\alpha}{2}(\mathbb{Z} + \tau\mathbb{Z})$, but are not contained in Λ . The function $\zeta \rightarrow \alpha^{-2} \wp_\tau\left(\frac{\zeta}{\alpha}\right)$ is uniquely determined by its periods, and by the fact that it is holomorphic on $\mathbb{C} \backslash \Lambda$, and has a pole with principal part $\zeta \rightarrow \frac{1}{\zeta^2}$ at 0. Hence the branch locus $\{\alpha^{-2} e_1(\tau), \alpha^{-2} e_2(\tau), \alpha^{-2} e_3(\tau)\}$ depends only on Λ , not on the specific choice of α and τ (though the respective ordered triple may depend on the specific choice of the parameters). Denote

$$BL_\Lambda = \{\alpha^{-2} e_1(\tau), \alpha^{-2} e_2(\tau), \alpha^{-2} e_3(\tau)\}.$$

We identify $\mathbb{C} \backslash \Lambda / \Lambda$ with the total space of the double branched covering (9.25). Extend it to the double branched covering of \mathbb{P}^1 with branch locus

$$(9.26) \quad \{\alpha^{-2} e_1(\tau), \alpha^{-2} e_2(\tau), \alpha^{-2} e_3(\tau), \infty\}.$$

Extend (9.20) to a holomorphic mapping of the closed torus \mathbb{C} / Λ onto the double branched covering of \mathbb{P}^1 with branch locus (9.26). Denote the double branched covering by DB_Λ . We associate to a smooth fiber bundle given in the form (9.7) the smooth map

$$(9.27) \quad X \ni z \rightarrow DB_{\Lambda(z)}.$$

Suppose \mathfrak{F} is a holomorphic elliptic fiber bundle over a finite open Riemann surface X given by a holomorphic mapping (9.19),

$$X \ni z \rightarrow \mathbb{C} / \Lambda(z).$$

Assign to each torus $\mathbb{C} / \Lambda(z)$ the branch locus $BL_{\Lambda(z)}$ of the double branched covering (9.25) of \mathbb{C} associated to $\mathbb{C} / \Lambda(z)$. Then $BL_{\Lambda(z)}$ depends holomorphically on z , since locally we can write $\Lambda(z) = \alpha(z)(\mathbb{Z} + \tau(z)\mathbb{Z})$ with α and τ holomorphic, the function $\wp_\tau(\zeta)$ depends holomorphically on τ and ζ and the points $\frac{1}{2}$, $\frac{\tau}{2}$ and

$\frac{\tau+1}{2}$ depend holomorphically on τ . Hence the locally defined functions $\alpha^{-2}e_j(\tau)$ depend holomorphically on τ and α .

For each lattice Λ the mapping $\mathbb{C} \ni \zeta \rightarrow -\zeta$ maps Λ onto itself. Hence it descends to an involution of \mathbb{C}/Λ , i.e. to a self-homeomorphism ι of \mathbb{C}/Λ such that $\iota^2 = \text{id}$.

Formula (9.21) for \wp_τ shows easily that

$$(9.28) \quad \left(\alpha^{-2} \wp_\tau \left(-\frac{\zeta}{\alpha} \right), \alpha^{-3} \wp'_\tau \left(-\frac{\zeta}{\alpha} \right) \right) = \left(\alpha^{-2} \wp_\tau \left(\frac{\zeta}{\alpha} \right), -\alpha^{-3} \wp'_\tau \left(\frac{\zeta}{\alpha} \right) \right).$$

In other words, the involution ι fixes the projection to \mathbb{P}^1 of the double branched covering and switches the sheets over each point. Hence, it fixes each of the four branch points and no other point.

Let $\tilde{\varphi}$ be any real linear self-homeomorphism \mathbb{C} which maps Λ onto itself, and let φ be the induced mapping on \mathbb{C}/Λ . Then φ commutes with ι . Indeed, $\tilde{\varphi}(-\zeta) = -\tilde{\varphi}(\zeta)$, $\zeta \in \mathbb{C}$.

Take any mapping class \mathbf{m} on \mathbb{C}/Λ . It contains a map that fixes a point. Lift the map to a self-homeomorphism of \mathbb{C} which maps Λ onto itself. The self-homeomorphism is isotopic to a real linear self-map $\tilde{\varphi}$ of \mathbb{C} which maps Λ onto itself. The induced map φ on \mathbb{C}/Λ commutes with ι . We saw that each mapping class \mathbf{m} on \mathbb{C}/Λ can be represented by a self-homeomorphism of \mathbb{C}/Λ which commutes with ι .

Let φ be a self-homeomorphism of \mathbb{C}/Λ which commutes with ι . Identify $\mathbb{C}\backslash\Lambda$ with the image of the branched covering (9.20). Denote the respective coordinates on the punctured torus by (u, v) . Write

$$\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v)).$$

By (9.28)

$$\varphi \circ \iota(u, v) = (\varphi_1(u, -v), \varphi_2(u, -v)) = (\varphi_1(u, v), -\varphi_2(u, v)) = \iota \circ \varphi(u, v).$$

Hence φ_1 depends only on the coordinate $u \in \mathbb{C}$, not on the sheet (determined by v). Further, if ι fixes (u, v) , then ι also fixes $\varphi(u, v)$. Hence φ maps the set of branch points BL_Λ onto itself, and maps ∞ to itself (since $\tilde{\varphi}$ fixes 0). Hence φ induces a self-homeomorphism of \mathbb{P}^1 in the class $\mathfrak{M}(\mathbb{P}^1; \infty, BL_\Lambda)$. (Note that the class is isomorphic to \mathcal{B}_3/Z_3 , the braid group modulo its center.) We denote the induced homeomorphism on \mathbb{P}^1 also by φ_1 and call it the projection $\varphi_1 = p(\varphi)$ of φ .

A lift of φ_1 to the double branched cover of \mathbb{P}^1 with branch locus $BL_\Lambda \cup \{\infty\}$ either coincides with φ or differs from φ by involution.

Consider now a smooth elliptic fiber bundle \mathfrak{F} over a finite Riemann surface X which is given by a smooth mapping (9.7). For each $z \in X$ we write the fiber as double branched covering $DB_{\Lambda(z)}$ over \mathbb{P}^1 . The mapping $X \ni z \rightarrow BL_{\Lambda(z)} \in C_3(\mathbb{C})/S_3$, which assigns to each $z \in X$ the branch locus over $\mathbb{C} \subset \mathbb{P}^1$ of the fiber, defines a smooth quasipolynomial of degree 3. We denote it by $f = P(\mathfrak{F})$ and call it the projection of the bundle \mathfrak{F} .

The following lemma holds.

LEMMA 9.4. 1) *Let \mathfrak{F} be an elliptic fiber bundle over a connected finite open Riemann surface X , given by a smooth mapping (9.7). If \mathfrak{F} is holomorphic then $f = P(\mathfrak{F})$ is holomorphic.*

2) Each smooth quasipolynomial f of degree 3 on X lifts to a smooth elliptic fiber bundle \mathfrak{F} on X , i.e. there exists an elliptic fiber bundle \mathfrak{F} such that $P(\mathfrak{F}) = f$.

Let \mathfrak{F}_1 and \mathfrak{F}_2 be two elliptic bundles over X given by smooth mappings of the form (9.7). Suppose that the projections $P(\mathfrak{F}_1)$ and $P(\mathfrak{F}_2)$ are isotopic quasipolynomial. Take a base point $z_0 \in X$. Then the monodromies of \mathfrak{F}_1 and \mathfrak{F}_2 at the base point z_0 along each generator of the fundamental group $\pi_1(X, z_0)$ differ by isotopy and conjugation or by the composition of the involution with isotopy and conjugation.

3) If f is holomorphic on X then f lifts to a holomorphic elliptic fiber bundle over X . Moreover, each isotopy class of lifts of f contains a holomorphic fiber bundle.

PROOF. Part 1) of the lemma has been proved. Prove part 2). Given a smooth (holomorphic, respectively) quasipolynomial f of degree 3 on X we will first construct a smooth (holomorphic, respectively) fiber bundle over X with fibers being punctured tori. The punctured tori over each $z \in X$ will be realized as double branched coverings of \mathbb{C} with branch locus $f(z)$. The elliptic fiber bundle is then obtained by considering the 1-point compactification of the fibers.

The following subset of $X \times \mathbb{C}^2$ with the induced structure is a smooth 4-manifold

$$(9.29) \quad \mathcal{X}^0 = \left\{ (z, \zeta, \zeta') \in X \times \mathbb{C}^2 : (\zeta')^2 = \prod_{\zeta_j \in f(z)} (\zeta - \zeta_j) \right\}.$$

It is easy to see that the projection $\mathcal{P} : \mathcal{X}^0 \rightarrow X$, $\mathcal{P}(z, \zeta, \zeta') = z$, is smooth. If f is holomorphic, then \mathcal{X} is a complex manifold and \mathcal{P} is holomorphic. Hence, $(\mathcal{X}^0, \mathcal{P}, X)$ is a smooth (holomorphic, respectively) fiber bundle with fibers being punctured tori.

Extend each fiber $\mathcal{P}^{-1}(z)$ of this bundle to a double branched covering over \mathbb{P}^1 with branch locus $f(z) \cup \{\infty\}$, $z \in X$. Denote the obtained set by \mathcal{X} . Smooth (holomorphic, respectively) coordinates on \mathcal{X} can be given as follows.

Put $\tilde{\zeta} = \frac{1}{\zeta}$ for $\zeta \in \mathbb{C} \setminus \{0\}$, $|\zeta|$ large and $\tilde{\zeta}' = \frac{\zeta}{\zeta'}$ for $\zeta, \zeta' \in \mathbb{C} \setminus \{0\}$, $\left| \frac{\zeta}{\zeta'} \right|$ small. The equation defining \mathcal{X}_0 becomes

$$(9.30) \quad (\tilde{\zeta}')^2 = \tilde{\zeta} \cdot \frac{1}{\prod_{\zeta_j \in f(z)} (1 - \tilde{\zeta}\zeta_j)}, \quad \tilde{\zeta} \neq 0, \quad \text{close to } 0.$$

\mathcal{X} is obtained by adding the points $(z, \tilde{\zeta}, \tilde{\zeta}') = (z, 0, 0)$. Compatible smooth (respectively holomorphic) coordinates on \mathcal{X} in a neighbourhood of these points are $(z, \tilde{\zeta})$. The projection, extended to \mathcal{X} is smooth (respectively holomorphic). We obtained a lift of the quasipolynomial f to an elliptic fiber bundle.

Let \mathfrak{F}_0 and \mathfrak{F}_1 be two smooth elliptic fiber bundles over X given in the form (9.7) such that the projections $f_0 = P(\mathfrak{F}_0)$ and $f_1 = P(\mathfrak{F}_1)$ are isotopic. Write the fibers of each \mathfrak{F}_j , $j = 0, 1$ as double branched coverings $DB_{\Lambda_j(z)}$ of \mathbb{P}^1 with branch locus $BL_{\Lambda_j(z)} \cup \{\infty\}$, and represent the bundles in the form (9.27).

We may isotop the bundle \mathfrak{F}_0 to a bundle $\tilde{\mathfrak{F}}_0$ through bundles given by (9.27) so that $P(\tilde{\mathfrak{F}}_0) = P(\mathfrak{F}_1)$. Indeed, the isotopy f_t , $t \in [0, 1]$, which joins f_0 and f_1 can be realized by using a continuous family of self-homeomorphisms of \mathbb{C} . Namely, there exists a continuous family $\psi_{t,z}$, $t \in [0, 1]$, $z \in X$, of self-homeomorphisms of

\mathbb{C} , $\psi_{t,z} : \mathbb{C} \rightarrow \mathbb{C}$, such that $f_t(z) = \psi_{t,z}(BL_{\Lambda_0(z)})$, and $\psi_{t,z}$ equals the identity outside a compact subset of \mathbb{C} .

Put

$$\mathcal{X}_t^0 = \left\{ (z, \zeta, \zeta') \in X \times \mathbb{C} : \zeta'^2 = \prod_{\zeta_j \in f_t(z)} (\zeta - \zeta_j), t \in [0, 1] \right\}.$$

Denote by $\mathcal{P}_t^0(z, \zeta, \zeta') = z$ the projection from \mathcal{X}_t^0 onto X . Let \mathcal{X}_t be obtained by extending each fiber of \mathcal{X}_t^0 to a double branched covering over \mathbb{P}^1 . Let \mathcal{P}_t be the extension of \mathcal{P}_t^0 to \mathcal{X}_t . Then the family $(\mathcal{X}_t, \mathcal{P}_t, X)$, $t \in [0, 1]$, is an isotopy of bundles of the form (9.27). Put $\tilde{\mathfrak{F}}_0 = (\mathcal{X}_1, \mathcal{P}_1, X)$. The bundles $\tilde{\mathfrak{F}}_0$ and $\tilde{\mathfrak{F}}_1$ have the same projection $P(\tilde{\mathfrak{F}}_0) = P(\tilde{\mathfrak{F}}_1)$. The fiber of both bundles over z_0 is the double branched covering of \mathbb{P}^1 with branch locus $BL_{\Lambda(z_0)}$. Take a simple closed curve in X with base point z_0 and let φ_0 , and φ_1 respectively, be the monodromies along this loop for the bundles $\tilde{\mathfrak{F}}_0$, and $\tilde{\mathfrak{F}}_1$ respectively.

After isotopies of each of the bundles (which preserve each common fiber) we may assume that φ_0 and φ_1 commute with involution. We obtain self-homeomorphisms φ_0 and φ_1 of the common fiber over z_0 such that their projections $p(\varphi_0)$ and $p(\varphi_1)$ coincide. Thus

$$(9.31) \quad p(\varphi_1 \varphi_2^{-1}) = p(\varphi_1) p(\varphi_2^{-1}) = \text{id}.$$

Hence, either $\varphi_1 = \varphi_2$ or $\varphi_1 \varphi_2^{-1} = \iota$, where ι is the involution in the fiber over z_0 . We proved 2).

Finally, suppose that f is holomorphic. We already proved that f has a holomorphic lift $\tilde{\mathfrak{F}}$. Choose any collection of circles of Q . Change the monodromy map of $\tilde{\mathfrak{F}}$ at z_0 along each circle of this collection by composing it with the involution and keep the previous monodromy map along each of the other circles. There is a holomorphic bundle over X which lifts f and has these monodromy maps. Indeed, consider the lift $\tilde{\mathfrak{F}} = (\tilde{U}\tilde{F}(X), \tilde{\mathcal{P}}, U\tilde{F}(X))$ of the bundle $\mathfrak{F} = (\mathcal{X}, \mathcal{P}, X)$. The transition maps from $\tilde{\mathcal{P}}^{-1}(\tilde{U}_j^-)$ to $\tilde{\mathcal{P}}^{-1}(\tilde{U}_j^+)$ are holomorphic for all j . For each circle in the chosen collection we change the respective transition map by composing it with a map $\tilde{\iota}_j$ which acts as involution in each fiber. The maps $\tilde{\iota}_j$ are holomorphic. For each choice of a collection of circles of Q we obtain a new holomorphic fiber bundle. The isotopy classes of the thus obtained bundles range over the isotopy classes of all bundles that lift \mathfrak{F} . The lemma is proved. \square

Let \mathfrak{F} be an elliptic fiber bundle over $\partial\mathbb{D}$ and let $\mathfrak{m}_{\mathfrak{F}}$ be the isotopy class of the monodromy map at the point 1. Suppose \mathfrak{F} is given by a map of the form (9.7) and let $f = P(\mathfrak{F})$ be the projection. f is a closed geometric braid. Let $\mathfrak{m}_{P(\mathfrak{F})}$ be the associated mapping class in the 3-punctured disc and let $\mathfrak{m}_{P(\mathfrak{F}),\infty} = \mathcal{H}_{\infty}(\mathfrak{m}_{P(\mathfrak{F})})$ be the associated mapping class in the 3-punctured complex plane. Notice that $\mathcal{H}_{\infty}(\mathfrak{m}_{P(\mathfrak{F})}) = p(\mathfrak{m}_{\mathfrak{F}})$, and hence for the conjugacy classes we have $\widehat{\mathfrak{m}_{P(\mathfrak{F}),\infty}} = \widehat{p(\mathfrak{m}_{\mathfrak{F}})}$. The right hand side depends only on $\widehat{\mathfrak{F}}$. We denote the right hand side also by $p(\widehat{\mathfrak{m}_{\mathfrak{F}}})$. Moreover, let \mathfrak{F}_1 be a bundle of the form (9.7) which is isotopic to \mathfrak{F} . Then the isotopy class $\mathfrak{m}_{P(\mathfrak{F}_1)}$ differs from $\mathfrak{m}_{P(\mathfrak{F})}$ by powers of a full twist. By Corollary 3.2 we have the following equality for the entropy of conjugacy classes

$$h(\widehat{p(\mathfrak{m}_{\mathfrak{F}})}) = h(\widehat{\mathfrak{m}_{P(\mathfrak{F}),\infty}}) = h(\widehat{\mathfrak{m}_{P(\mathfrak{F})}}),$$

and the right hand side does not change if the bundle \mathfrak{F} is replaced by an isotopic bundle \mathfrak{F}_1 .

The following proposition says that the conformal module of an isotopy class $\widehat{\mathfrak{F}}$ of elliptic fiber bundles over the circle is equal to the conformal module of the conjugacy class of the closed braid $P(\mathfrak{F})$ for any bundle \mathfrak{F} representing $\widehat{\mathfrak{F}}$. Moreover, the proposition relates the entropy of the monodromy mapping class of the elliptic bundle to the entropy of the projection of the class.

PROPOSITION 9.1. *Let $\widehat{\mathfrak{F}}$ be a free isotopy class of elliptic fiber bundles over the circle $\partial\mathbb{D}$ and let the bundle \mathfrak{F} be a representative of $\widehat{\mathfrak{F}}$. Let $\widehat{\mathfrak{m}}_{\mathfrak{F}}$ be the associated monodromy mapping class and let $p(\widehat{\mathfrak{m}}_{\mathfrak{F}})$ be its projection. Suppose $\varphi_0 \in \widehat{\mathfrak{m}}_{\mathfrak{F}}$ is a self-homeomorphism of the fiber over 1 which lifts to a real linear self-map of \mathbb{C} . Then*

$$(9.32) \quad h(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = h(\varphi_0) = h(p(\varphi_0)) = h(p(\widehat{\mathfrak{m}}_{\mathfrak{F}})) ,$$

and

$$(9.33) \quad \mathcal{M}(\widehat{\mathfrak{F}}) = \mathcal{M}(P(\mathfrak{F}))$$

The following analog of Theorem 1 is an immediate Corollary.

COROLLARY 9.1.

$$h(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = \frac{\pi}{2} \frac{1}{\mathcal{M}(\widehat{\mathfrak{F}})} .$$

PROOF. Indeed, by Theorem 1

$$\mathcal{M}(P(\mathfrak{F})) = \frac{\pi}{2} \frac{1}{h(\widehat{\mathfrak{m}}_{P(\mathfrak{F})})} = \frac{\pi}{2} \frac{1}{h(p(\widehat{\mathfrak{m}}_{\mathfrak{F}}))} .$$

The corollary follows from (9.32) and (9.33). \square

Proof of Proposition 9.1. We may assume that the fiber over 1 is $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. The lift $\tilde{\varphi}_0$ of φ_0 corresponds to a matrix

$$(9.34) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) .$$

If the trace $\mathrm{tr}(A) = a + d$ equals zero then an easy calculation shows that the fourth power of the matrix is the identity. Hence, the mapping $\tilde{\varphi}_0$, and therefore also the mapping φ_0 , is periodic. Thus, $h(\varphi_0) = 0$ and therefore $h(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = h(\varphi_0)$ in this case.

If $|a + d| > 1$ then either both eigenvalues of A are positive or both eigenvalues are negative, and their product equals one.

Suppose A has positive eigenvalues. Denote by λ the largest eigenvalue. By the example in [1] we have $h(\varphi_0) = \log \lambda$. Moreover, consider φ_0 as self-homeomorphism of a torus with one distinguished point. Introduce a conformal structure on the torus by the mapping onto the standard torus which lifts to a linear self-map of \mathbb{C} with the following properties. It maps the eigenvector v_1 of A corresponding to λ to the unit vector in the real direction and maps the other eigenvector v_2 to the unit vector in the imaginary direction. Conjugate φ_0 with this conformal structure. The conjugated map preserves the canonical quadratic differential dz^2 and hence is absolutely extremal. Since entropy is a conjugacy invariant we obtain $h(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = h(\varphi_0)$.

In the remaining case $(-I)A$ has positive eigenvalues and for the involution ι the lift of $\iota \circ \varphi_0$ corresponds to a matrix with positive eigenvalues. Again, $h(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = h(\varphi_0)$.

Suppose $|a + d| = 1$. Possibly after composing φ_0 with the involution ι we may assume that $a + d = 1$. Then A has multiple eigenvalue 1 and is either the identity or is conjugate to the matrix

$$(9.35) \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The columns of the conjugating matrix B consist of an eigenvector of A and the associated vector. If A is the identity the entropy of φ_0 equals zero. In the remaining case $\tilde{\varphi}_0$ is conjugate to a real linear self-map $\tilde{\varphi}_1$ of \mathbb{C} which maps a lattice $B(\mathbb{Z} + i\mathbb{Z}) = v_1\mathbb{Z} + v_2\mathbb{Z}$ onto itself. Here B is the conjugating matrix (and has real coefficients), and v_1 and v_2 are the generators of the lattice (written as complex numbers). The induced map φ_1 on the quotient $\mathbb{C}/(v_1\mathbb{Z} + v_2\mathbb{Z})$ is conjugate to φ_0 .

The real linear mapping $\tilde{\varphi}_1$ equals

$$\tilde{\varphi}_1(xv_1 + yv_2) = (x + y)v_1 + yv_2.$$

The mapping

$$xv_1 + yv_2 \rightarrow e^{\frac{2\pi i}{v_1}(xv_1 + yv_2)} = w$$

maps \mathbb{C} onto \mathbb{C}^* . The torus $\mathbb{C}/(v_1\mathbb{Z} + v_2\mathbb{Z})$ can be written as $\mathbb{C}^*/(w \sim e^{\frac{2\pi i v_2}{v_1}} w)$. On the closure of a fundamental domain for the covering

$$\mathbb{C}^* \rightarrow \mathbb{C}^*/(w \sim e^{\frac{2\pi i v_2}{v_1}} w)$$

the mapping $\tilde{\varphi}_1$ induces a twist

$$e^{\frac{2\pi i}{v_1}(xv_1 + yv_2)} = w \rightarrow w e^{2\pi i y}.$$

By Lemma 3.7 $h(\varphi_0) = h(\varphi_1) = 0$, hence again $h(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = h(\varphi_0)$.

By Lemma 3.1 we have $h(p(\varphi_0)) = h(\varphi_0)$. If $h(\varphi_0) = 0$ then $h(p(\widehat{\mathfrak{m}}_{\mathfrak{F}})) \leq h(p(\varphi_0)) = 0 = h(\varphi_0)$. Otherwise, φ_0 is pseudo-Anosov, in other words, it is absolutely extremal as self-homeomorphism of the torus with a distinguished point. If $\psi \in p(\widehat{\mathfrak{m}}_{\mathfrak{F}})$ is an absolutely extremal self-homeomorphism of \mathbb{P}^1 with distinguished points, then it lifts to an absolutely extremal self-homeomorphism ψ_0 of the torus with distinguished point, hence

$$h(p(\widehat{\mathfrak{m}}_{\mathfrak{F}})) = h(\psi) = h(\psi_0) = h(\widehat{\mathfrak{m}}_{\mathfrak{F}}) = h(\varphi_0).$$

The equations (9.32) are proved.

Prove equation (9.33). Let A be any annulus of conformal module $m(A) < \mathcal{M}(\widehat{\mathfrak{F}})$. The isotopy class $\widehat{\mathfrak{F}}$ can be represented by a holomorphic elliptic fiber bundle \mathfrak{F} on A which is given by a holomorphic mapping of the form (9.19). Then $f = P(\mathfrak{F})$ is a holomorphic quasipolynomial on A whose monodromy map at 1 represents the class $\widehat{\mathfrak{m}}_{f,\infty} = \widehat{\mathfrak{m}}_{P(\mathfrak{F}),\infty} = p(\widehat{\mathfrak{m}}_{\mathfrak{F}})$. Hence,

$$\mathcal{M}(\widehat{P(\mathfrak{F})}) \geq \mathcal{M}(\widehat{\mathfrak{F}}).$$

Vice versa, take an annulus A of conformal module $m(A) < \mathcal{M}(\widehat{P(\mathfrak{F})})$. There exists a holomorphic quasipolynomial f of degree 3 on A whose monodromy map at 1

represents $p(\widehat{\mathfrak{m}}_{\mathfrak{F}})$. By Lemma 9.4 there is a holomorphic elliptic fiber bundle \mathfrak{F} on A which lifts f and is in the class $\widehat{\mathfrak{F}}$. Hence

$$\mathcal{M}(\widehat{\mathfrak{F}}) \geq \mathcal{M}(\widehat{\mathcal{P}(\mathfrak{F})}).$$

Proposition 9.1 is proved. \square

We will now prove Theorem 3. Suppose $(\mathcal{X}^*, \mathcal{P}^*, X)$ is a holomorphic \mathbb{C}^* -bundle over a Riemann surface X . Note that a group G of fiber preserving biholomorphic maps of \mathcal{X}^* with free and properly discontinuous action is generated by a single element which acts in each fiber by multiplication with a complex number of absolute value different from one.

Indeed, the restriction of each $g \in G$ to each \mathbb{C}^* -fiber $(\mathcal{P}^*)^{-1}(x)$ is a holomorphic self-map of the fiber. A holomorphic self-map of \mathbb{C}^* is multiplication by a complex number $\lambda \neq 0$ (depending on g, x and on the choice of coordinates on \mathbb{C}^*), followed, maybe, by inversion $\zeta \rightarrow \frac{1}{\zeta}$, $\zeta \in \mathbb{C}^*$. The requirement that G acts freely and properly discontinuously on fibers implies automatically the following facts. For each $g \in G$ and for each $x \in X$ we have $|\lambda| \neq 1$. Moreover, G does not contain the inversion.

By the same reason G is generated by a single element. Indeed, for $|\lambda| \neq 0, 1$ the quotient $\mathbb{C}^*/(\zeta \sim \lambda\zeta)$ is compact, so for $|\lambda'| \neq 0, 1$ the action of multiplication by λ' on the quotient has either a fixed point or a limit point.

Proof of Theorem 3. Let S be a smooth torus with a hole and let $\widehat{\mathfrak{F}}$ be an isotopy class of smooth oriented elliptic fiber bundles on S . For each bundle $\mathfrak{F} \in \widehat{\mathfrak{F}}$ which is given by a smooth mapping

$$(9.36) \quad S \ni x \rightarrow \mathbb{C}/\Lambda(x) \cong DB_{\Lambda(x)}.$$

we consider the projection $f = \mathcal{P}(\mathfrak{F})$ which is a smooth separable quasipolynomial of degree 3 on X .

Let $w_j : S \rightarrow X_j$, $j = 1, \dots, 4$, be the four complex structures on S chosen in Theorem 8.1. Suppose the push-forward of $\widehat{\mathfrak{F}}$ to each X_j contains a holomorphic bundle \mathfrak{F} . Then the push-forward of $f = \mathcal{P}(\mathfrak{F})$ to each X_j is isotopic to a holomorphic quasipolynomial.

By Theorem 2 the quasipolynomial f is isotopic to a holomorphic quasipolynomial for each conformal structure of second kind on S . By Lemma 9.4 the elliptic fiber bundle is isotopic to a holomorphic elliptic bundle for each conformal structure of second kind on S .

By Theorem 8.1 the free isotopy class of f corresponds to the conjugacy class of a homomorphism

$$\Phi : \pi_1(S, x_0) \rightarrow \Gamma \subset \mathcal{B}_3,$$

where Γ is a subgroup of \mathcal{B}_3 .

Suppose $f = \mathcal{P}(\mathfrak{F})$ is an irreducible quasipolynomial. By Theorem 8.1 we may assume that Γ is generated by $\sigma_1\sigma_2$. Consider the associated mapping class \mathfrak{m} in the 3-punctured disc. Take its image $\mathfrak{m}_\infty = \mathcal{H}_\infty(\mathfrak{m})$ in the mapping class of the 3-punctured complex plane. The conjugacy class $\widehat{\mathfrak{m}_\infty}$ may be represented by the mapping φ which acts on \mathbb{C} with set of distinguished points $\left\{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\right\}$ by rotation around the origin by the angle $\frac{2\pi}{3}$. Denote by the same letter φ the

extension to \mathbb{P}^1 . Lift \mathbb{P}^1 with set of distinguished points $E = \left\{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, \infty\right\}$, to be double branched covering of \mathbb{P}^1 with branch locus E . Consider a lift $\tilde{\varphi}$ of φ to a self-homeomorphism of this double branched covering.

After a free isotopy of f we may assume that $f(x_0) = E$ and the image under \mathcal{H}_∞ of the monodromy map of f at the base point z_0 along the first generator equals φ^{k_1} and the respective map for the second generator equals φ^{k_2} . Here k_1 and k_2 are integral numbers. After an isotopy we may assume that the original bundle \mathfrak{F} is a lift of f and the monodromy maps of the bundle \mathfrak{F} are $\tilde{\varphi}_1 = \iota^{\ell_1} \varphi^{k_1}$ and $\tilde{\varphi}_2 = \iota^{\ell_2} \varphi^{k_2}$, where each ℓ_j , $j = 1, 2$, equals either 0 or 1.

A holomorphic elliptic fiber bundle on a closed torus with these monodromy maps is obtained as follows. Consider any closed torus $X^c = \mathbb{C}/\Gamma$, where Γ is a lattice, $\Gamma = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$, with complex structure induced by \mathbb{C} . Let S be smoothly embedded into X^c so that the image of a loop representing the first generator of the fundamental group of S is lifted to the segment $[0, \lambda_1]$ and a respective loop for the second generator is lifted to the segment $[0, \lambda_2]$.

Let the Riemann surface Y be the double branched covering of \mathbb{P}^1 with branch locus $E = \left\{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, \infty\right\}$. Consider $Y \setminus \{\infty\}$ as image of the embedding (9.20) into \mathbb{C}^2 . Denote the coordinates in \mathbb{C}^2 by (ζ, ζ') .

The lattice $\Gamma = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$ acts on $\mathbb{C} \times Y$ as follows. We associate to the element $\lambda_1 n + \lambda_2 m \in \Gamma$ the mapping

$$(9.37) \quad (z, (\zeta, \zeta')) \rightarrow \left(z + \lambda_1 n + \lambda_2 m, \left(\zeta \cdot e^{\frac{2\pi i}{3} k_1 n} \cdot e^{\frac{2\pi i}{3} k_2 m}, \zeta' \cdot (-1)^{\ell_1 n} \cdot (-1)^{\ell_2 m} \right) \right).$$

The trivial bundle $\mathbb{C} \times Y \rightarrow \mathbb{C}$ descends to a holomorphic fiber bundle

$$(9.38) \quad (\mathbb{C} \times Y)/\Gamma \rightarrow \mathbb{C}/\Gamma$$

with fiber Y and monodromies $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$. The bundle is isotrivial. Indeed, the 6th power of each monodromy map is the identity. Hence, an unramified covering of the bundle of degree 12 is the trivial bundle.

We proved Theorem 3 for the case when the projection $P(\mathfrak{F})$ of a fiber bundle $\mathfrak{F} \in \widehat{\mathfrak{F}}$ is an irreducible quasipolynomial, and obtained that in this case option 1 occurs.

Suppose f is reducible. By Theorem 8.1 the group Γ is generated either by a conjugate of $\sigma_1 \sigma_2 \sigma_1$ or by a conjugate of σ_1 together with Δ_3^2 . The first case corresponds to a periodic mapping class. By the same arguments as above the first part of Theorem 3 holds and the extended bundle over the closed surface is isotrivial.

Suppose, after conjugation, that Γ is generated by σ_1 and Δ_3^2 . Prove the theorem in this case. Let $\mathbf{m}(\sigma_1)$ and $\mathbf{m}(\Delta_3^2)$ be the mapping classes on the 3-punctured disc associated to σ_1 and Δ_3^2 . Denote by $\mathbf{m}(\sigma_1)_\infty = \mathcal{H}_\infty(\mathbf{m}(\sigma_1))$, $\mathbf{m}(\Delta_3^2)_\infty = \mathcal{H}_\infty(\mathbf{m}(\Delta_3^2))$ the images in the mapping class group of the 3-puncture plane. Notice that $\mathbf{m}(\Delta_3^2)_\infty = \text{id}$. Hence, after conjugation we may assume that Γ corresponds to the subgroup of $\mathfrak{M}(\mathbb{C}; \emptyset, \{-1, 1, 3\})$, which is generated by the mapping class of a positive half-twist about the segment $[-1, 1]$. Let φ represent the positive half-twist.

A lift of the mapping class $\mathbf{m}(\sigma_1)_\infty$ to a mapping class on the punctured torus can be described as follows. Consider the double branched covering $\mathbf{p} : \mathring{Y} \rightarrow \mathbb{C}$ of \mathbb{C} with branch locus $\{-1, 1, 3\}$. Take a lift $\tilde{\varphi}$ of φ . Extend $\tilde{\varphi}$ to a self-map of the double branched covering Y of \mathbb{P}^1 with branch locus $\{-1, 1, 3, \infty\}$ denoted again by $\tilde{\varphi}$. The extension of \mathbf{p} to Y is denoted as before by \mathbf{p} .

Let $\gamma = \mathbf{p}^{-1}([0, 1])$ be the preimage on Y of the segment $[-1, 1]$ under the covering map \mathbf{p} . γ is a closed curve on Y which represents a generator of the fundamental group of Y . The mapping $\tilde{\varphi}$ is a Dehn twist about γ , possibly followed by involution. To see this consider an arc in \mathbb{C} which intersects the segment $[-1, 1]$ transversally and its lift to the double branched covering and look at the action of $\tilde{\varphi}$ on the lifted arc.

Write Y as quotient \mathbb{C}/Λ for a lattice Λ and describe the mapping class of the Dehn twist in terms of the quotient \mathbb{C}/Λ . We consider a Dehn twist about a curve γ representing a generator of the fundamental group. Hence, after multiplying the lattice by a non-zero complex number we may assume that $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ for a complex number τ and γ lifts to the segment $[0, 1]$ in \mathbb{C} . The mapping class of this Dehn twist on \mathbb{C}/Λ can be represented by a self-homeomorphism ψ_τ of \mathbb{C}/Λ which lifts to the real linear self-map $\tilde{\psi}_\tau$ of \mathbb{C} which maps 1 to 1 and τ to $1 + \tau$. This can be seen by looking at the action of $\tilde{\psi}_\tau$ on a curve that lifts a representative of a second generator of the fundamental group of \mathbb{C}/Λ . Note that the real 2×2 matrix corresponding to $\tilde{\psi}_\tau$ is $\begin{pmatrix} 1 & (\operatorname{Im} \tau)^{-1} \\ 0 & 1 \end{pmatrix}$.

Let X be a Riemann surface of second kind and let $w : S \rightarrow X$ be a conformal structure on S . Denote by \mathfrak{F}_X a holomorphic bundle on X which is isotopic to the push-forward of \mathfrak{F} , has fiber Y over $z_0 = w(x_0)$, and has the form $X \ni z \rightarrow \mathbb{C}/\Lambda(z)$. The monodromy mappings of the bundle at the point $z_0 \in X$ are isotopic to $\iota^{\ell_1} \tilde{\psi}_\tau^{k_1}(z_1^-)$ and $\iota^{\ell_2} \tilde{\psi}_\tau^{k_2}(z_2^-)$ for numbers ℓ_j which are equal to 0 or to 1, $j = 1, 2$. The lifts of these mappings are real linear self-maps of \mathbb{C} which map 1 to $(-1)^{\ell_j}$ and τ to $(-1)^{\ell_j}(\tau + k_j)$, and hence, map $\mathbb{Z} + \tau\mathbb{Z}$ onto itself.

Let $UF(X)$, U_j and $\tau(\tilde{z}), \tilde{z} \in UF(X)$, have the previous meaning. Then for $j = 1, 2$, $z_j \in U_j$, we have $\tau(z_j^+) = \tau(z_j^-) + k_j$. Consider the holomorphic bundle defined by $UF(X) \ni \tilde{z} \rightarrow \mathbb{C}/\tilde{\Lambda}(\tilde{z})$, where $\tilde{\Lambda}(\tilde{z}) = \mathbb{Z} + \tau(\tilde{z})\mathbb{Z}$. For $j = 1, 2$ and $z_j \in U_j$ we have

$$\tilde{\Lambda}(\tilde{z}_j^-) = \mathbb{Z} + \tau(\tilde{z}_j^-)\mathbb{Z} = \mathbb{Z} + (\tau(\tilde{z}_j^-) + k_j)\mathbb{Z} = \mathbb{Z} + \tau(\tilde{z}_j^+)\mathbb{Z} = \tilde{\Lambda}(\tilde{z}_j^+).$$

Hence $\tilde{\Lambda}$ descends to a holomorphic mapping $z \rightarrow \Lambda(z)$ on X . The mapping $X \ni z \rightarrow \mathbb{C}/\Lambda(z)$ defines a holomorphic elliptic bundle. The monodromy maps of this bundle at z_0 are isotopic to $\tilde{\psi}_\tau^{k_1}$ and $\tilde{\psi}_\tau^{k_2}$. Suppose $\ell_j = 1$ for one of the j or for both j . Then we change the bundle $X \ni z \rightarrow \mathbb{C}/\Lambda(z)$ by changing the gluing map over the respective U_j to the composition of the previous gluing map with the involution ι . The thus obtained bundle is holomorphic and isotopic to \mathfrak{F}_X . Keep the previous notation \mathfrak{F}_X for the new bundle.

Map the quotient $\mathbb{C}/(\zeta \sim \zeta + 1)$ to \mathbb{C}^* by the exponential mapping $\zeta \rightarrow e^{2\pi i \zeta} = w$. Each fiber $\mathbb{C}/(\mathbb{Z} + \tau(\tilde{z})\mathbb{Z})$, $\tilde{z} \in UF(X)$, is isomorphic to $\mathbb{C}^*/\{w \sim we^{2\pi i \tau}\}$. Put $\lambda(\tilde{z}) = e^{2\pi i \tau(\tilde{z})}$. Notice that $\lambda(\tilde{z}_j^+) = \lambda(\tilde{z}_j^-)$ so that we obtain a holomorphic

function on X which we also denote by λ . We have $|\lambda(z)| \neq 1$ since $\text{Im } \tau \neq 0$, and $\lambda(z) \neq 0$.

If the numbers ℓ_j , $j = 1, 2$, in the definition of the monodromy maps of the bundle \mathfrak{F}_X are zero then the bundle \mathfrak{F}_X is isomorphic to the quotient of the trivial bundle $X \times \mathbb{C}^*$ by the action of the group generated by the mapping

$$g(z, \zeta) = (z, \lambda(z) \zeta), \quad (z, \zeta) \in X \times \mathbb{C}^*.$$

If $\ell_j = 1$ for one or both j then the respective monodromy mapping is isotopic to a mapping that lifts to a real linear self-map of \mathbb{C} which maps 1 to -1 and $\tau(\tilde{z}_j^-)$ to $-(\tau(\tilde{z}_j^-) + k_j)$. In this case we consider the \mathbb{C}^* -bundle over X which is obtained from the trivial \mathbb{C}^* -bundle over $UF(X)$ by using the transition map $(z, \zeta) \rightarrow (z, \frac{1}{\zeta})$ rather than the identity on the respective U_j . The bundle \mathfrak{F}_X is obtained as quotient with respect to the action of the group generated by the mapping g which lifts to the bundle over $UF(X)$ as multiplication of the variable in the fiber over \tilde{z} by the number $\lambda(z)$. The mapping g acts on points in U_j in coordinates (\tilde{z}_j^-, ζ) by $(\tilde{z}_j^-, \zeta) \rightarrow (\tilde{z}_j^-, \lambda(z) \zeta)$ and in coordinates $(\tilde{z}_j^+, \tilde{\zeta}) = (\tilde{z}_j^+, \frac{1}{\zeta})$ by $(\tilde{z}_j^+, \frac{1}{\zeta}) \rightarrow (\tilde{z}_j^+, \frac{1}{\lambda(z) \zeta})$.

We obtain that in this case option 2 occurs. Since the monodromy maps commute the bundle extends to a smooth bundle on the closed torus. Since the projection $f = P(\mathfrak{F})$ is not isotopic to a holomorphic quasipolynomial for any conformal structure of first kind unless f is isotrivial, the bundle \mathfrak{F} is not isotopic to a holomorphic elliptic bundle for any conformal structure of first kind unless the bundle is isotrivial.

Theorem 3 is proved. □

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